K-THEORY AND DAG

1. K-THEORY

1.1. ∞-categories.

(1) Reminder on simplicial stuff
(2) Definition of ∞-categories (quasicategories)
(3) Functors
(4) Adjunctions
(5) Mapping spaces
(6) ∞-category of spaces
(7) Yoneda
(8) Homotopy category
(9) Equivalences in ∞-categories
(10) Equivalences between ∞-categories
(11) Maximal subgroupoid
(12) Geometric realization
(13) Limit constructions

1.2. Stable ∞-categories.

(1) Definition of stable ∞-categories [Lur09, Lur16a]
(2) Triangulated structure on the homotopy category
(3) Stabilization of ∞-categories
(4) Example: derived category of abelian category
(5) K_0 of a stable ∞-category

1.3. Spectra. See also [Gep19], [Lur16a], [YD19]

(1) Definition of spectra and ring spectra
(2) Stabilization
(3) Mapping spectra in stable ∞-categories
(4) Compact objects
(5) $\mathcal{M}od_R, \mathcal{M}od_R^{\mathrm{perf}}, \mathcal{M}od_R^{\mathrm{proj}}$ for ring spectra R
(6) Connection to (simplicial) commutative rings

1.4. $S_\bullet$-construction and K_0.

(1) Waldhausen $S_\bullet$ construction on pointed ∞-category [Lur14, §14]
(2) Example: K-theory of ring spectra via projective modules and via perfect modules [Lur14, §19]
(3) Relation to K_0; K_0 by hand
(4) Relation to $BGL(A)^+$
(5) Example: K_0(finite pointed spaces)

1.5. Reduction to K-theory of spectra. See also [BGT10]

(1) Additivity theorem [Lur14, §17]
(2) Invariance under stabilization
(3) Ind-objects and idempotent completion [Lur14, §15], [Lur06]
(4) $K(\mathrm{Ind}(\mathcal{C}))$ for stable ∞-categories $\mathcal{C}$

Date: January 23, 2020.

(5) K-theory of pointed ∞-categories with finite colimits in terms of K-theory of ring spectra $\mathrm{End}(C)$ [Lur14, §19]

1.6. **Additive and Localizing invariants.**

(1) Presentable ∞-cats
(2) Verdier quotients of presentable stable ∞-categories
(3) (Split) exact sequences of stable ∞-categories
(4) Additive invariants [BGT10]
(5) K-theory is the universal additive invariant

1.7. **Non-connective K-theory.**

(1) Localizing invariants
(2) Non-connective K-theory
(3) Universal property
(4) K-theory of open immersion of schemes
(5) Weibel's conjecture
(6) Cofinality theorem and connection to connective K-theory

1.8. **Connections to classical K-theory.**

(1) Exact 1-categories (Or Waldhausen, depending on what is easier to connect to the previous)
(2) K-theory of exact 1-categories
(3) Theorem of the heart
(4) Classical results: Excision, Mayer-Vietoris, ... [Wei13, §5]

2. DAG

2.1. **Higher topos theory.**

(1) Grothendieck topologies
(2) ∞-sites
(3) Left-exact localization functors
(4) Groupoid objects
(5) ∞-topoi
(6) Example of ∞-topoi: ∞-sheaves on ∞-sites
(7) ∞-stacks on $\mathrm{Sh}(T)$ are equivalently ∞-sheaves on T, for ∞-topoi T

2.2. **Derived schemes.** We can follow [Kha16], [Mat12]

(1) Derived rings, the associated E_∞-rings, the underlying discrete rings
(2) Open immersions of derived affine schemes
(3) Zariski topology on $\mathrm{s}\mathcal{R}\mathrm{ing}$
(4) Open immersions of Zariski stacks
(5) Derived schemes

2.3. **Derived Stacks.** We can follow [Kha16], [Mat12]

(1) Cotangent complex
(2) Étale topology
(3) Derived étale stacks
(4) Quasi-coherent modules on stacks: connective and non-connective, the case of derived schemes
(5) Smooth morphisms of derived schemes
(6) Derived Artin stacks

2.4. **Perfect complexes and K-theory of stacks.**

(1) Quasi-coherent sheaves on stacks, Artin stacks, schemes
(2) The adjunction $f^* \dashv f_*$ for a map of stacks $f : \mathcal{X} \to \mathcal{Y}$
(3) Perfect complexes on derived stacks [BZFN10], [Lur16b], Khan lecture notes
(4) Symmetric monoidal ∞-categories: dualizable objects
(5) Compact objects
(6) The relations between perfect complexes, dualizable object and compact object in $\mathrm{QCoh}(X)$
(7) f^* preserves perfect complexes
(8) $\mathcal{P}\mathrm{erf}(X)$ is stable ∞-cat

2.5. **K-theory of projective bundles.**

(1) Locally free modules of finite rank
(2) Projective bundles on stacks and Serre's twisting sheaf [Kha18b, §6]
(3) Semi-orthogonal decompositions of stable ∞-categories [Kha18b], [YD19]
(4) Additive invariants 'split' semi-orthogonal decompositions [Kha18b, Lem. 2.3.3]
(5) Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$

2.6. **K-theory of blow-ups.**

(1) Quasi-smooth immersions of derived Artin stacks
(2) Derived blow-ups in quasi-smooth centers
(3) Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$

2.7. **Zariski Descent.** See [Kha18b, §3,4]

(1) Quasi-compact open immersions of derived schemes
(2) qcqs derived schemes
(3) Perfect complexes that vanish on open subsets
(4) Zariski descent for squares
(5) Zariski descent for Čech covers

2.8. **Grothendieck-Riemann-Roch I.** See [Kha18a, §9]

(1) Derived Sym and Λ
(2) Additive K-theory (earlier?)
(3) λ-rings, γ-operations, associated grading
(4) λ-ring structure on $K_0(R) = K_0^{\oplus}(\mathcal{M}\mathrm{od}_R^{\mathrm{proj}})$
(5) Global resolution property
(6) Globalization of λ-ring structure to $K_0(X)$ for derived scheme X

2.9. **Grothendieck-Riemann-Roch II.** See [Kha18a, §10]

(1) Formulation of GRR-theorem for derived schemes

References

[BGT10] Andrew J. Blumberg, David Gepner, and Goncalo Tabuada. A universal characterization of higher algebraic k-theory, 2010.

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and drinfeld centers in derived algebraic geometry. *Journal of the American Mathematical Society*, 23(4):909–966, 2010.

[Gep19] David Gepner. An introduction to higher categorical algebra. *arXiv preprint arXiv:1907.02904*, 2019.

[Kha16] Adeel Khan. *Motivic homotopy theory in derived algebraic geometry*. PhD thesis, 2016.

[Kha18a] Adeel Khan. The Grothendieck–Riemann–Roch theorem . *Lecture notes*, 2018.

[Kha18b] Adeel A. Khan. Descent by quasi-smooth blow-ups in algebraic k-theory, 2018.

[Lur06] J. Lurie. *Higher Topos Theory*. Princeton University Press, August 2006.

[Lur09] Jacob Lurie. Derived algebraic geometry I: stable ∞-categories. *Preprint. http://arxiv.org/abs/math/0608228*, 2009.

[Lur14] Jacob Lurie. Algebraic K-Theory and Manifold Topology. *Lecture notes*, 2014.

[Lur16a] Jacob Lurie. Higher algebra. 2014. *Preprint, available at http://www. math. harvard. edu/˜ lurie*, 2016.

[Lur16b] Jacob Lurie. Spectral algebraic geometry. *Preprint, available at www. math. harvard. edu/˜ lurie/papers/SAG-rootfile. pdf*, 2016.

[Mat12] Akhil Mathew. Simplicial commutative rings, 2012.

[Wei13] Charles A Weibel. *The K-book: An introduction to algebraic K-theory*, volume 145. American Mathematical Society Providence, RI, 2013.

[YD19] Lucy Yang and Jack Davies. Algebraic K-theory - European Talbot 2019 notes . *Lecture notes*, 2019.