Spectra, Ring spectra and their categories of modules

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1 Spectra

As mentioned in the previous talk, if we take $\mathcal{S}$ the ∞-category of spaces, then we can define the category of spectra Sp as the stabilization $Stab(\mathcal{S})$ of $\mathcal{S}$ with respect to the loop functor Ω. The corresponding formula is

$$Sp = \lim(\ldots \to \mathcal{S}_* \to \mathcal{S}_* \to \mathcal{S}_*)$$

where all the maps are the loop functor.

We will present here another description of spectra, which turns out to be equivalent to the one above, but which is closer to the way spectra are usually introduced. We will define the spectra as a certain full subcategory of a category whose objects are called *prespectra*. Before we start, we should mention that the objects we will call spectra here are sometimes called Ω-*spectra*, and the role of spectra under this other convention is here played by our prespectra.

Definition 1.1. Let $\mathcal{C}$ be a pointed ∞-category. A *prespectrum object* of $\mathcal{C}$ is a functor $X\colon N(\mathbb{Z}\times\mathbb{Z}) \to \mathcal{C}$, where $\mathbb{Z}\times\mathbb{Z}$ is endowed with its natural structure of poset, with the following property: for every pair of integers $i \neq j$, the value $X(i,j)$ is a zero object of $\mathcal{C}$. We denote by $PSp(\mathcal{C})$ the full subcategory of $Fun(N(\mathbb{Z}\times\mathbb{Z}), \mathcal{C})$ spanned by prespectrum objects.

We see that the interesting objects are the objects $X(n,n)$, for n an integer. However the other objects are necessary to obtain the full datum of a prespectrum. In particular it follows from the definition above that for every $n \in \mathbb{Z}$ there exist maps $\Sigma X(n,n) \to X(n+1,n+1)$ and $X(n,n) \to \Omega X(n+1,n+1)$. We can distinguish different types of interesting prespectra depending on properties of those maps, but for this talk we will only define *spectra*:

Definition 1.2. The category $Sp(\mathcal{C})$ of spectra objects in $\mathcal{C}$ is the full subcategory of $PSp(\mathcal{C})$ spanned by those prespectra for which all maps $X(n,n) \to \Omega X(n+1,n+1)$ are homotopy equivalences.

The map $(i,j) \mapsto (i+1, j+1)$ induces by precomposition a functor $S\colon PSp(\mathcal{C}) \to PSp(\mathcal{C})$ called the *shift functor*. In particular we have the formula $SX(n,n) = X(n+1,n+1)$.

Proposition 1.3. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. Then:*

1. *For every object $X \in Sp(\mathcal{C})$, the canonical map $X \to \Omega_{PSp(\mathcal{C})}S(X)$ is an equivalence.*

2. *The shift functor $S\colon Sp(\mathcal{C}) \to Sp(\mathcal{C})$ is a homotopy inverse to $\Omega_{Sp(\mathcal{C})}$.*

3. *The ∞-category $Sp(\mathcal{C})$ is stable.*

For this talk, the most interesting result of previous proposition is the third one, stating that the ∞-category $\mathcal{C}$ that we just constructed is a stable one.

2 Ring spectra

A *commutative ring spectrum* is a commutative algebra object in the symmetric monoidal ∞-category of spectra. This definition sounds very easy, but if we want to understand ring spectra we need to understand the different terms in the above definition. We will first introduce *symmetric monoidal ∞-categories*, and then describe their *commutative algebra objects*. Alternatively one could define such objects via operads and cocartesian fibrations, and in the end we would end up with equivalent categories. There is however in this context a simpler formulation, and this is the one we will be using.

In this talk we will focus on commutative ring spectra because they admit a simpler description. It is however possible to provide a similar description for associative ring spectra and work with them instead.

Definition 2.1. We introduce the $(2,1)$ category $Span(Fin)$ of finite sets and spans between them. More specifically, its objects are finite sets, its 1-morphisms from I_0 to I_1 are spans $I_0 \leftarrow J_0 \to I_1$, and its 2-morphisms are (iso)morphisms $J_0 \to J_1$ making the losange-shaped diagram commute. In this category, the product of two finite sets I_0, I_1 is simply their disjoint union.

From here, we can define the category of commutative monoid objects in any ∞-category $\mathcal{C}$ with finite products as the category of product-preserving functors

$$CMon(\mathcal{C}) := Fun_\pi(Span(Fin), \mathcal{C}).$$

In particular, for $\mathcal{C} = Cat_\infty$ the category of (small) infinity categories, we get that $CMon(Cat_\infty)$ is the category of symmetric monoidal ∞-categories. In other words, a symmetric monoidal ∞-category is an object of $CMon(Cat_\infty)$, or equivalently a functor $\Delta^0 \to CMon(Cat_\infty)$. Note also that this setting allows to define a symmetric monoidal functor between ∞-categories as a 1-morphism in $CMon(Cat_\infty)$, or equivalently as a functor $\Delta^1 \to CMon(Cat_\infty)$.

Example 2.2. We will use that the category Fin of finite sets is a symmetric monoidal category (under the disjoint union) so it is nice to see here how one can realize Fin as a symmetric monoidal category in this setting:

We define Fin as the functor $Fin\colon Span(Fin) \to Cat$, which sends a finite set A to the overcategory Fin/A, and which sends a span $A \leftarrow A' \to B$ to the functor $Fin/A \to Fin/A' \to Fin/B$ defined by $(X \to A) \mapsto (X \times_A A' \to A') \mapsto (X \times_A A' \to A' \to B)$.

Example 2.3. Let us illustrate the definition above in the case of the usual commutative monoids (i.e. commutative monoid objects in the category of sets, viewed as an ∞-category whose higher order morphisms are all trivial). Let M be a monoid. For any finite set S of cardinality n, we send S to $F(S) = M^n$. Then if we have a span $[n] \leftarrow [m] \rightarrow [p]$ with $f\colon [n] \leftarrow [m]$ and $g\colon [m] \rightarrow [p]$, we define

$$\Delta_f\colon F([n]) \rightarrow F([m]) \text{ via } \Delta_f(a_1, a_2, \ldots, a_n) = (a_{f(1)}, \ldots, a_{f(m)})$$

and

$$\Sigma_g\colon F([m]) \rightarrow F([p]) \text{ via } \Sigma_g(b_1, \ldots b_m) = \left(\sum_{g(i)=j} b_i \right)_{j=1,\ldots,p}.$$

Thus the full span is sent to the map $(\Sigma_g \circ \Delta_f)\colon F([n]) \rightarrow F([p])$

The 2-morphisms encode properties of the multiplication. For example it is easy to define a 2-morphism encoding the commutativity of our monoid, i.e. inducing an isomorphism between xy and yx for x, y arbitrary elements of our monoid.

It still remains to define algebra objects in a symmetric monoidal ∞-category M. In this setup, it is simply introduced as a symmetric monoidal functor $Fin \rightarrow M$. Note that in this definition the category $Span(Fin)$ seems to have vanished, but it is actually required in order to define the datum associated to a symmetric monoidal functor.

Remark 2.4. As a sanity check, we can remark that when $\mathcal{C}$ is the nerve of a 1-category, our definition is actually equivalent to the usual definition of a commutative monoid.

3 Modules over ring spectra

Let $\mathcal{R}$ be a (commutative) ring spectrum, as seen previously we can see $\mathcal{R}$ as an algebra object over the category of spectra. In this section we will implicitly restrict the discussion to commutative ring spectra, but everything said here can also be done for any associative ring spectrum. We will present in the appendix one way to construct the category of modules over $\mathcal{R}$. Alternatively we can define the module category over $\mathcal{R}$, denoted $Mod_{\mathcal{R}}$, via the machinery of ∞-operads; both definitions turn out to be equivalent. We will introduce two important subcategories $Mod_{\mathcal{R}}^{perf}$ and $Mod_{\mathcal{R}}^{proj}$ consisting of respectively the perfect modules and the projective modules, but first of all we need to introduce the notion of compact objects in a category:

Definition 3.1. Let $\mathcal{C}$ be an ordinary category that admits filtered colimits. An object $X \in \mathcal{C}$ is called compact (or finitely presented/presentable) if the corepresentable functor

$$Hom_{\mathcal{C}}(X, -)\colon \mathcal{C} \rightarrow \text{Set}$$

preserves filtered colimits. In other words, for every filtered category $\mathcal{I}$ and every functor $F\colon \mathcal{I} \rightarrow \mathcal{C}$, the morphism

$$\varinjlim_i \mathcal{C}(X, F(i)) \rightarrow \mathcal{C}(X, \varinjlim_i F(i))$$

is an isomorphism.

Remark 3.2. The above definition translates literally to a definition of compact objects in an ∞-category:

Let $\mathcal{C}$ be an ∞-category that admits filtered colimits. An object $X \in \mathcal{C}$ is called compact if the corepresentable functor

$$Hom_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \text{Spaces}$$

preserves filtered colimits.

One can define a filtered ∞-category $\mathcal{C}$ as having the property that for every finite poset A, $\mathcal{C}$ has the right lifting property with respect to the inclusion $N(A) \subset N(A \cup \{\infty\})$. In particular every filtered ∞-category is a sifted ∞-category, and if $\mathcal{C}$ is filtered, then the diagonal functor $\Delta \colon \mathcal{C} \to \mathcal{C} \times C$ is cofinal. A filtered colimit is a colimit for which the indexing category is filtered.

Let R be a ring spectrum. An R-module M is *perfect* if M is a compact object of $Mod_{\mathcal{R}}$.

Definition 3.3. The category $Mod_{\mathcal{R}}^{perf}$ is the full subcategory of $Mod_{\mathcal{R}}$ spanned by the perfect modules. Alternatively, we can define $Mod_{\mathcal{R}}^{perf}$ as the smallest full subcategory of $Mod_{\mathcal{R}}$ which contains $\mathcal{R}$ and is closed under finite colimits, desuspensions and passage to direct summands.

Example 3.4. We can associate to a ring R its Eilenberg-MacLane spectrum HR and the objects of the associated category of modules Mod_R are equivalent to chain complexes of R-modules. Then the objects of Mod_R^{perf} can be identified with bounded chain complexes of finitely generated projective R-modules. This result is due to Schwede-Shipley

Both categories $Mod_{\mathcal{R}}$ and $Mod_{\mathcal{R}}^{perf}$ are stable, so in particular $Mod_{\mathcal{R}}^{perf}$ is a pointed ∞-category which admits finite colimits, and we can consider its K-theory.

Definition 3.5. Let $\mathcal{R}$ be an associative ring spectrum. We set $K(\mathcal{R}) = K(Mod_{\mathcal{R}}^{perf})$, and we will refer to $K(\mathcal{R})$ as the *algebraic K-theory space* of $\mathcal{R}$.

For $\mathcal{R}$ a ring spectrum and M an $\mathcal{R}$-module, we say that M is *finitely generated and projective* if it can be realized as a direct summand of $\mathcal{R}^n$ for some n. Then $Mod_{\mathcal{R}}^{proj}$ will denote the full subcategory of $Mod_{\mathcal{R}}$ spanned by the finitely generated projective $\mathcal{R}$-modules. Note in particular that $Mod_{\mathcal{R}}^{proj}$ is a full subcategory of $\mathcal{R}^{perf}$. This category has finite coproducts (but in general not finite colimits), so we can define its additive K-theory $K_{add}(Mod_{\mathcal{R}}^{proj})$.

Theorem 3.6. *Let $\mathcal{R}$ be a* connective *ring spectrum (i.e. $\pi_n(\mathcal{R}) = 0$ for $n < 0$). Then the inclusion $Mod_{\mathcal{R}}^{proj} \to Mod_R^{perf}$ induces a homotopy equivalence of K-theory spaces*

$$K_{add}(Mod_{\mathcal{R}}^{proj}) \to K(Mod_{\mathcal{R}}^{perf}) = K(R).$$

4 Simplicial commutative rings

Definition 4.1. Let R be a commutative ring. We let $Poly_R$ denote the category whose objects are polynomial rings $R[x_1, \dots, x_n]$ over R, and whose morphisms are R-algebra homomorphisms. The category $Poly_R$ admits finite coproducts created by the tensor product. We let $CAlg_R^{\Delta}$ denote the full subcategory $Fun^{\pi}(Poly_R^{op}, \mathcal{S}) \subseteq Fun(Poly_R^{op}, \mathcal{S})$ spanned by functors preserving finite products, where $\mathcal{S}$ denotes the category of spaces. We refer to $CAlg_R^{\Delta}$ as the *∞-category of simplicial commutative R-algebras*. When $R = \mathbb{Z}$ we drop the index R and call $CAlg^{\Delta}$ the *∞-category of simplicial rings*.

One may of course wonder why such objects are called simplicial. This is explained by the following remark:

Remark 4.2. Let A_R be the ordinary category of simplicial commutative R-algebras, regarded as a simplicial model category in the usual way, and let A° denote the full subcategory spanned by the fibrant-cofibrant objects. Then there is a canonical equivalence of ∞-categories $N(A^\circ) \to CAlg_R^{\Delta}$, obtained via the Yoneda embedding $B \mapsto \underline{Hom}(-, B)$

Remark 4.3. Let R be a commutative ring. The ∞-category $CAlg_R^{\Delta}$ can be characterized up to equivalence by the following properties:

1. The ∞-category $CAlg_R^{\Delta}$ is presentable.
2. There exists a coproduct-preserving, fully faithful functor $j\colon Poly_R \to CAlg_R^{\Delta}$.
3. The essential image of j consists of compact projective objects of $CAlg_R^{\Delta}$ which generate $CAlg_R^{\Delta}$ under sifted colimits.

A sifted colimit is a colimit for which the indexing category is sifted. We say that an ∞-category $\mathcal{C}$ is sifted if the diagonal map $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is a cofinal functor. We already mentioned before that filtered categories are sifted. Another important example of a sifted category is the category Δ^{op}.

Proposition 4.4. *Let R be a commutative ring and let $j\colon Poly_R \to CAlg_R^{\Delta}$ denote the Yoneda embedding. Let $\mathcal{C}$ be an ∞-category which admits small sifted colimits, and $Fun_\Sigma(CAlg_R^{\Delta}, \mathcal{C})$ the full subcategory of $Fun(CAlg_R^{\Delta}, \mathcal{C})$ spanned by the functors preserving sifted colimits. Then:*

1. *Composition with j induces an equivalence of ∞-categories $Fun_\Sigma(CAlg_R^{\Delta}, \mathcal{C}) \to Fun(Poly_R, \mathcal{C})$.*
2. *A functor $F\colon CAlg_R^{\Delta} \to \mathcal{C}$ belongs to $Fun_\Sigma(CAlg_R^{\Delta}, \mathcal{C})$ if and only if F is a left Kan extension of $F \circ j$ along j.*
3. *Suppose $\mathcal{C}$ admits finite coproducts, and let $F\colon CAlg_R^{\Delta} \to \mathcal{C}$ preserve sifted colimits. Then F preserves finite coproducts if and only if $F \circ j$ preserves finite coproducts.*

Let R be a commutative ring and let $Poly_R$ denote the category of finitely generated polynomial rings over R, which we regard as a full subcategory of both the ∞-category $CAlg_R^{\Delta}$ of simplicial commutative R-algebras and the ∞-category $CAlg_R^{cn}$ of connective E_∞-algebras in chain complexes over R. There is an essentially unique functor

$$\Theta\colon CAlg_R^{\Delta} \to CAlg_R^{cn}$$

which commutes with small sifted colimits and restricts to the identity on $Poly_R$. If A is an object of $CAlg_R^{\Delta}$, we will denote $\Theta(A)$ by A° and refer to it as the *underlying E_∞-algebra of A*.

Proposition 4.5. *Let R be a commutative ring, and let $\Theta\colon CAlg_R^{\Delta} \to CAlg_R^{cn}$ be the forgetful functor defined above. Then:*

1. *The functor Θ preserves small limits and colimits.*
2. *The functor Θ is conservative.*
3. *If R contains the field $\mathbb{Q}$, then Θ is an equivalence of ∞-categories.*

We define the homotopy groups and the K-theory of a simplicial commutative R-algebra as the homotopy groups and K-theory of the underlying E_∞ object. We define similarly its category of modules.

Theorem 4.6. *Let R be a simplicial commutative ring. Then the canonical homomorphism $K_0(R) \to K_0(\pi_0(R))$ is bijective.*

A Describing the category of modules over a commutative ring spectrum

We can of course define algebra objects and modules over them using the full machinery of ∞-operads and cocartesian fibrations, and this would give the same objects as the ones we are defining here. Since we are working in a very specific setting we can however simplify the machinery in order to get to a situation where it is easier to visualize our constructions. Similarly to what we did in the case of the algebra objects, the key part of our definition will be a symmetric monoidal functor from a certain category to the category in which live our algebra objects (in our case, this category is the category of spectra). Unfortunately the category serving as the domain of our functor is a bit more complicated than the one for algebra objects:

Definition A.1. Let $\mathcal{E}$ be the category of finite sets with marked elements. More precisely, the objects are pairs (I, M) with I a finite set and M a subset of I (it is not required that M is a proper subset of I), and the morphisms $(I_0, M_0) \to (I_1, M_1)$ are the functions $f\colon I_0 \to I_1$ whose restriction to M_0 is a bijective function $f|_{M_0}\colon M_0 \to M_1$. This category comes with a structure of symmetric monoidal functor with the tensor product

$$(I_0, M_0) \otimes (I_1, M_1) = (I_0 \coprod I_1, M_0 \coprod M_1).$$

We will study the category $\mathcal{M} := Fun^{s.m.}(\mathcal{E}, \mathcal{C})$ of symmetric monoidal functors from $\mathcal{E}$ to $\mathcal{C}$ (we can choose $\mathcal{C}$ to be the category of spectra, but the arguments work with any symmetric monoidal ∞-category). We will see that we can consider this category as the category of pairs (A, X) with A an algebra object in $\mathcal{C}$ and X a module over A.

Definition A.2. There is a symmetric monoidal functor $i\colon Fin \to \mathcal{E}$ sending the finite set I to the pair $(I, \emptyset)$. This functor induces a functor $\mathcal{M} \to Fun^{s.m.}(Fin, \mathcal{C})$ and we say that this map sends an object in $\mathcal{M}$ to its *underlying commutative algebra*. In particular we can see the category of modules over an algebra A as the fiber of this map over the algebra A (recall that the codomain is the category of algebra objects in $\mathcal{C}$).

For every algebra object A, we want that A is a left module over itself. In our setting, this would be implied by a map $Fun^{s.m.}(Fin, \mathcal{C}) \to \mathcal{M}$ such that composition with the above defined map gives the identity. This map should be induced by a symmetric monoidal functor $\mathcal{E} \to Fin$, and the obvious choice in this situation is the forgetful functor $(I, X) \mapsto I$.

Definition A.3. There is a functor $\{pt\} \to \mathcal{E}$ given by $\{pt\} \mapsto (\{pt\}, \{pt\})$ (note that this functor is not required to be symmetric monoidal), and this functor induces a map $\mathcal{M} \to Fun(\{pt\}, \mathcal{C}) = Ob(\mathcal{C})$, which should be understood as projecting the pair (A, X) to X. We say that this map sends an object in $\mathcal{M}$ to its *underlying module*.

Remark A.4. There are two special objects in $\mathcal{E}$, the pair $(\{pt\}, \emptyset)$ and the pair $(\{pt\}, \{pt\})$. All different objects in $\mathcal{E}$ can be formed by disjoint unions of these two objects. Moreover, maps into either of these two objects can encode some operations on algebras and modules:

- If we have a map $(I, M) \to (\{pt\}, \emptyset)$, then necessarily $M = \emptyset$ and this map can be interpreted as a map $A^{\otimes n} \to A$ with n the cardinality of I and A an algebra object.

- If we have a map $(I, M) \to (\{pt\}, \{pt\})$, then $M = \{i\}$ for some element $i \in I$ and a functor $F \colon \mathcal{E} \to \mathcal{C}$ encodes a homotopy from this map to a map $A^{\otimes n-1} \otimes X \to A \otimes X \to X$ with n the cardinality of I, where the first map is the algebra structure on A, and the second map is the structure of left module over A on X.