INFINITY-CATEGORIES

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Abstract. Category theory is an useful and powerful way of dealing with information about mathematical objects. In the sense of category theory we talk about two mathematical objects as "the same" if they are isomorphic. However, there are many situations where a weaker notion of "the same" is more useful, not only when working with categories themselves as the mathematical objects of interest. To deal with this, one introduces the concept of higher categories. In this talk, we focus on $(\infty, 1)$-categories, which we will abusively refer to simply as ∞-categories. There are many equivalent models to describe these, and we will here take the approach of quasicategories, simplicial sets with certain lifting properties, as developed by Joyal and Lurie. In this model, we show how to make sense of many categorical notions, including, but not limited to, functors, natural transformations, adjunctions, and equivalences in and between categories.

1. How and why ∞-categories?

Category theory is an useful and powerful way of dealing with information about mathematical objects and morphisms between them. In the sense of category theory we talk about two mathematical objects x and y as "the same" if they are **isomorphic**: as in, if we can find morphisms $f : x \to y$ and $g : y \to x$ such that $gf = \mathrm{id}_x$ and $fg = \mathrm{id}_y$. Indeed, many important results in mathematics deal with presicely this sort of "classfication of objects up to isomorphism". However, there are many situations a weaker notion of "the same" is more useful, as illustrated in the following few examples.

Example 1.1. Two topological spaces X and Y are homotopy equivalent if there is a homotopy equivalence between them. As in, if we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that gf and fg are homotopic to id_X and id_Y, respectively, in the sense that we can find a continuous map $H : X \times I \to X$ such that $H(x, 0) = (gf)(x)$ and $H(x, 1) = (\mathrm{id}_X)(x)$, and similarly for fg.

Example 1.2. Even in category theory itself we do not care about categories up to isomorphism, but rather, up to equivalence. Given two categories $\mathscr{C}$ and $\mathscr{D}$ it is too strict to ask for $\mathscr{C}$ and $\mathscr{D}$ to be isomorphic in the sense that we can find functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that GF and FG are the identity functors "on the nose". Rather, we only require them to be naturally isomorphic to the identities, in the sense that we can find a natural isomorphism $\tau : GF \to \mathrm{id}_{\mathscr{C}}$, and similarly for FG.

Note that both of these examples involved some notion of morphisms between morphisms: homotopies or natural transformation. Collectively, we refer to these as 2-morphism and say that the above examples organise into 2-categories. We could envision also having access to morphisms between 2-morphisms, what we would then call 3-morphisms, and so on. This allows us to envision n-categories as mathematical structures where we have access to k-morphisms for all $k \leq n$. Of course, as in ordinary categories (which in this language should be referred to as 1-categories) we should also have access to some way of composing k-morphisms whenever their source and targets match up. Finally, we envision ∞-categories as being some sort of limit of this where we have access to n-morphisms for all positive n. This is all very hand-wavy and not so rigorously defined, but gives us a good idea of what sort of theory we are looking for[1]. In particular, one example that we want to fit into this formalism is the following.

Example 1.3. Let X be a topological space. We can extract an n-category $\tau_{\leq n}X$, called **the fundamental n-groupoid**, from X in the following manner. Objects in $\tau_{\leq n}X$ are points in X, morphisms are paths $[0, 1] \to X$ between points, 2-morphisms are homotopies $[0, 1]^2 \to X$ between two such paths, 3-morphisms are homotopies $[0, 1]^3 \to X$ between homotopies, and so on. Finally, we regard two n-morphisms $[0, 1]^n \to X$ as the same if they are homotopic to one another. Let us figure our what these look like for some small n:

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[1]A axiomatation of precisely what we want from higher categories is given in [Toë05].

- If $n = 0$, then we are looking at all the points of our space X and considering two such point to be equal if there is a path between them. Hence, the fundamental 0-groupoid is the set of path-components of X.
- If $n = 1$, we have an ordinary category with objects being the points and the morphisms being homotopy classes of paths. Note in particular that this is a groupoid; all the morphisms are invertible since we can walk a path backwards (recall that two 1-morphisms are the same if they are homotopic to one another). Classically, this groupoid is referred to as the **fundamental groupoid** of the space X.

Note that even in the higher cases, these guys are "groupoids" in that all morphisms are invertible in this higher categorical sense.

In general, working with higher categories is difficult stuff. However, it is still useful to consider the significantly simplified situation where we assume that many of our k-morphisms are invertible. Let us say that an (n, k)-category is a structure with i-morphisms for all $0 \leq i \leq n$ such that i-morphisms are invertible whenever $i > k$. With this terminology we can place our examples in the following table:

(n,k)	0	1	2
0	$\tau_{\leq 0}X$	$\tau_{\leq 1}X$	$\tau_{\leq 2}X$
1		Category	Example 1.1
2			Example 1.2

In this talk we will focus on $(\infty, 1)$-categories, which we will just abusively call **∞-categories** from now on. Likewise, we will refer to $(\infty, 0)$-categories as **∞-groupoids**. In particular, note that the fundamental ∞-groupoid of a space X is in fact a ∞-groupoid in this sense. In fact, one of the guiding principles of higher category theory is that all ∞-groupoids are of this form. The hypothesis that ∞-groupoids are the same thing as spaces, or more precisely homotopy types, is referred to as **Grothendieck's homotopy hypothesis**. In the quasicategory model for ∞-categories, which we will review soon, we interpret Theorem 3.2 as a proof of this fact.

2. Quasi-categories

Summarizing the discussion in the previous section: we want ∞-categories to be a theory that contains both topological spaces and categories as special cases. There are very many different equivalent models for ∞-categories, just as there are different models for the homotopy category of CW-complexes, denoted $\mathcal{H}$ here. The brute force solution is to mash the two requirements together and consider topological categories, that is, categories enriched over topological spaces. In practice this approach to ∞-categories is impossible to work with; there are too many technical difficulties. A more sophisticated solution to our problem is to consider simplical sets.

2.1. Simplicial sets.

Recall that the **simplex category** Δ is the category whose objects are the ordered sets $[n] = \{0, 1, 2, \cdots, n\}$ and whose morphisms are order-preserving maps between these.

Definition 2.1. The category of **simplicial sets** is the functor category

$$\mathrm{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}).$$

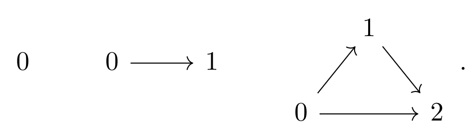
For a simplicial set X we write $X_n = X([n])$ and refer to the elements in this set as the **n-simplicies** of X. We denote by

$$\Delta^n = \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

the presheaf on Δ represented by $[n]$, and refer to this as the **standard n-simplex**. We can use the Yoneda lemma to see that

$$X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X).$$

This allows us to think of an n-simplex of X both as an element in the set X_n and as a map $x : \Delta^n \to X$ of simplicial sets. Often, we visualise the first three standard n-simplices as the diagrams



Similarly, low-dimensional simplicies in X are depicted by replacing the numbers with whereever these elements are sent in X.

Remark 2.2. We can in particular single out the morphisms

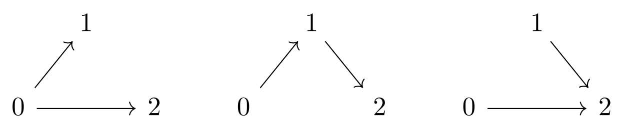
$$\partial_k^n : [n-1] \to [n] \quad \text{and} \quad \sigma_k^n : [n+1] \to [n]\,,$$

in the simplex category Δ. The first one is the unique injective map that does not hit the value k, and the second one is the unique surjective map that hits k twice. These generate Δ is a suitable sense. Using these, we can combinatorially specify a simplicial set as a sequence of sets X_n together with maps

$$d_k^n = (\partial_k^n)^* : X_n \to X_{n-1} \quad \text{and} \quad s_k^n = (\sigma_k^n)^* : X_n \to X_{n+1}\,,$$

called **faces** and **degeneracies**, respectively, satisfying certain simplicial relations. This is the classical approach found in for example [May67], and we encourage the reader to check for themselves that this definition agrees with the one we have given above. We emphasize that the combinatorial behaviour of simplicial sets makes them extremely practical to work with in many situations.

The standard n-simplex has some important subcomplexes called horns. The kth horn of Δ^n, denoted Λ_k^n, is the subcomplex generated by all faces except for the face opposite to the vertex numbered k. The two horns Λ_0^n and Λ_n^n are referred to as the **outer n-horns**, while the other n-horns are referred to as the **inner n-horns**. The three 2-horns Λ_0^2, Λ_1^2, and Λ_2^2 are depicted below:



The category of simplicial sets is a cartesian symmetric monoidal category. The function objects are given as

$$\mathrm{Map}_{\mathrm{sSet}}(X, Y) = \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta^\bullet, Y)\,.$$

2.2. **Topological spaces and Kan complexes.** We have a good supply of simplicial sets coming from topological spaces. There is an adjunction[2] between topological spaces and simplicial sets:

$$|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

The left adjoint $|-| : \mathrm{sSet} \to \mathrm{Top}$ is called the **geometric realization functor** and always yields a CW-complex [GJ99, Proposition I.2.3]. The right adjoint is the functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ that assigns a topological space X to the simplicial set $\mathrm{Sing}(X)$ defined as

$$[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X).$$

This is referred to as the **singular complex** of X. Singular complexes of topological spaces are simplicial sets exhibiting a certain specific property: they are all Kan complexes.

Definition 2.3. A **Kan complex** K is a simplicial set where every horn $\Lambda_k^n \to K$ can be extended to a simplex $\Delta^n \to K$.

We note that this gives us an indication of how we can view simplicial sets geometrically. For instance, a 0-simplex $v : \Delta^0 \to X$ can be interpreted as a point of X, a 1-simplex $\phi : \Delta^1 \to X$ as a path in X from the point $x = d_1(e)$ to the point $y = d_0(e)$. Note that this goes further though, we can view a 2-simplex $\Delta^2 \to X$ as a homotopy between paths, and so on for higher simplicies.

[2]The adjunction we have specified above is not only an adjunction; it is actually a Quillen equivalence in the sense of model category theory. In particular, this tells us that an equivalent model for the homotopy category of CW-complexes is the homotopy category of Kan complexes.

2.3. **Nerves of categories.** We also have a nice supply of simplicial sets coming from ordinary categories. Recall that posets can be regarded as a full subcategory of Cat in a natural way, which in particular provides us with a fully faithful inclusion functor $i : \Delta \to \mathrm{Cat}$. The nerve funcor is defined as evaluation at i in the sense that:

$$N : \mathrm{Cat} \to \mathrm{sSet}\,, \quad \mathscr{C} \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(-), \mathscr{C})\,.$$

This provides us with a fully faithful functor. The reader can check for themself that the 0-simplicies in $N\mathscr{C}$ are given by the objects in $\mathscr{C}$ and that the 1-simplicies in $N\mathscr{C}$ are given by the morphisms in $\mathscr{C}$. We can let this serve as a paradigm when thinking of a simplicial set X as category-like:

- Let us refer to a 0-simplex $v : \Delta^0 \to X$ as an object of X.
- Let us refer to a 1-simplex $\phi : \Delta^1 \to X$ as a morphism of X.

From this point of view it is natural to refer to $x = d_1(\phi)$ and $y = d_0(\phi)$ as the source and target of the morphism ϕ, respectively. If we want to emphasize this point of view of simplicial sets encoding category theory we will write $\phi : x \to y$ for our 1-simplex. Similarly, it is natural to write $\mathrm{id}_v = s_0(v) : v \to v$, and refer to this as the identity on v

Proposition 2.4. *Let X be a simplicial set. The following conditions are equivalent:*

(1) There is a small category $\mathscr{C}$ such that $X \simeq N(\mathscr{C})$.
(2) Every inner n-horn $\Lambda^n_k \to X$ can be uniquely extended to an n-simplex $\Delta^n \to X$.

Proof. See [Lur09, Proposition 1.1.2.2]. □

There is also a groupoid statement; we would require unique extensions of all horns in that case, so that the resulting simplicial set is a Kan complex. Indeed, the nerve of a category is a Kan complex if and only if the category is a groupoid.

2.4. **Quasi-categories.** We can take Proposition 2.4 as a motivation for the following definition, which we hope will give a good model for ∞-categories. In the higher categorical setting it would make sense to get rid of the uniqueness for the lifts, though.

Definition 2.5. A **quasicategory** is a simplicial set in which every inner horn $\Lambda^n_k \to X$ can be extended to a simplex $\Delta^n \to X$.

It is immediately clear that Kan complexes and nerves of categories are quasicategories. Hence, topological spaces and categories are special cases of the definition via the singular complex functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ and the nerve functor $N : \mathrm{Cat} \to \mathrm{sSet}$, respectively. If we use the category-like description of simplicial sets, the inner 2-horn condition tells us that any pair of composable morphisms has a composite. Indeed, any pair of composable morphisms $\phi, \psi : \Delta^1 \to X$ determines an inner 2-horn $\Lambda^2_1 \to X$, informally visualised as

$$\begin{array}{ccccc} & & d_0(\phi) = d_1(\psi) & & \\ & \overset{\phi}{\nearrow} & & \overset{\psi}{\searrow} & \\ d_1(\phi) & & \cdots\cdots\cdots\cdots\rightarrow & & d_0(\psi) \end{array} .$$

The inner 2-horn condition tells us that the composite, the dotted map in the diagram, exists. While this composite is not unique in a strict sense, the rest of the inner horn conditions guarantee that it is unique in a higher categorical sense. Indeed, possible compositions of ϕ and ψ form a simplicial set; the pullback in the diagram

$$\begin{array}{ccc} \mathrm{Comp}_X(\phi,\psi) & \longrightarrow & \mathrm{Map}_{\mathrm{sSet}}(\Delta^2, X) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(\phi,\psi)} & \mathrm{Map}_{\mathrm{sSet}}(\Lambda^2_1, X) \end{array} .$$

The condition that we have lifts for all inner horns guarantees that this simplicial set is actually a Kan complex, and even more, that this Kan complex is contractible. If we want to emphasize this higher categorical view of uniqueness we usuaully speak of something "being unique up to contractible choice"[3].

[3] A way to rigourously state this sort of uniquement of compositions as the characterizing feature of a quasicategory is given by the following result.

Remark 2.7. For emphasis: quasicategories is a way to interpret the language of ∞-categories. For conceptual reasons we have kept quasicategories and ∞-categories as distinct in this talk, but it is very common to refer to quasicategories as ∞-category. Indeed, this is what Lurie does in [Lur09]. However, there are technically many more equivalent ways of interpreting ∞-categories; quasicategories is just a model that has proven to be quite useful for practical purposes. In particular, Lurie uses the equivalence between quasicategories and simplicial categories extensively throughout [Lur09]. See also [Ber18] for a monograph on various models of ∞-categories and how they compare to one another.

3. Concepts related to quasicategories

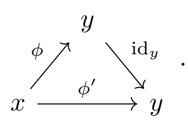
The following section deals with various concepts related to ∞-categories, such as the homotopy category, quasigroupoids, mapping spaces, as well as examples of quasicategories not arising as the nerve of a category or as the singular complex of a space.

3.1. **The homotopy category.** The nerve functor $N : \mathrm{Cat} \to \mathrm{sSet}$ admits a left adjoint $h : \mathrm{sSet} \to \mathrm{Cat}$. We refer to hX as the **homotopy category** of X. Note that the homotopy category functor is the unique colimit preserving functor that extends $\Delta \to \mathrm{Cat}$. Unfortunately, colimits of categories are ugly, so the category hX is a bit tricky to describe in general. However, it is possible to get a fairly tangible description when the simplicial set is a quasicategory. See [Lur09, Section 1.2.3] for details.

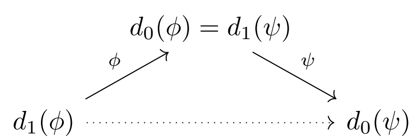
- The objects of the category hX are the 0-simplicies $\Delta^0 \to X$.
- Recall the category point of view paradigm of simplicial sets in which we denote 1-simplices as morphisms $\phi : x \to y$. We say that two morphisms $\phi, \phi' : x \to y$ are **equivalent** if there is a 2-simplex $\sigma : \Delta^2 \to X$ such that

  $$d_0(\sigma) = \mathrm{id}_y\,, \quad d_1(\sigma) = \phi'\,, \quad d_2(\sigma) = \phi\,.$$

  Visually, think of σ being the diagram

  The check that this is indeed an equivalence relation on the 1-simplicies $\phi : x \to y$ in X uses that we have inner 3-horn lifting in our quasicategory, by definition. We will denote the equivalence class of ϕ by $[\phi]$, in standard fashion. The set of morphisms between x and y in the homotopy category hX is the set of equivalences classes of morphisms $\phi : x \to y$.
- Now we need to figure out how to compose to classes $[\phi]$ and $[\psi]$, whenever they are composable. This of course means that their source and target match up; more explicitly that $d_1(\psi) = d_0(\phi)$. Note that each such pair determines an inner horn $\Lambda^2_1 \to X$, informally visualised as:

  Because of the filling of inner horns we know that this can be lifted to a 2-simplex $\sigma : \Delta^2 \to X$. We define the composition between $[\phi]$ and $[\psi]$ as

  $$[\psi] \circ [\phi] = [d_1 \sigma]\,.$$

  The check that this is well-defined again uses liftings for inner 3-horns.

Theorem 2.6 (Joyal)**.** *A simplicial set X is a quasicategory if and only if the restriction map* $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ *is an acyclic Kan fibration.*

We can think of $\mathrm{Map}(\Lambda^2, X)$ as "the space composition problems in X" and of $\mathrm{Map}(\Delta^2, X)$ as the "the space of solutions to composition problems". The theorem above tells us that the characterizing property of a quasicategory is that these two spaces are the same, from a homotopical point of view.

3.2. **Quasigroupoids and Kan complexes.** One of the guiding principles of higher category theory was the homotopy hypothesis; that ∞-groupoids were supposed to be the same things as homotopy types. With quasicategories as our model for ∞-categories we first need to decide on a model for ∞-groupoids. As a preliminary definition, let us say that morphism $\phi : \Delta^1 \to X$ in a quasicategory is called an **equivalence** if its image $[\phi]$ in the homotopy category hX is an isomorphism.

Definition 3.1. A quasicategory X is called a **quasigroupoid** if all morphisms X are equivalences, or in other words, if the homotopy category hX is a groupoid in the ordinary sense.

Since (homotopy types of) spaces correspond to (homotopy types of) Kan complexes it makes sense to formulate the homotopy hypothesis into the following statement, which is the main result of the article [Joy02].

Theorem 3.2 (Joyal)**.** *A quasicategory is an quasigroupoid if and only if it is a Kan complex.*

Inside every quasi-category we can always find the so-called **maximal quasigroupoid**; the largest subcomplex containing only 1-simplicies that are equivalences. Formally, this is the pullback in the diagram

$$\begin{array}{ccc} X^{\simeq} & \longrightarrow & X \\ \downarrow & & \downarrow \\ N(hX^{\cong}) & \longrightarrow & NhX \end{array}$$

of simplicial sets. Here $hX^{\cong}$ denotes the ordinary maximal groupoid of the category hX.

3.3. **Mapping spaces.** In the same way as the mapping objects in categories are sets we want the mapping objects of ∞-categories to be ∞-groupoids. Another fundamental thing that we want from our ∞-categories is for the mapping objects between ∞-categories to be ∞-groupoids. In this model this would mean that we want mapping objects to be Kan complexes.

Definition 3.3. Let $x, y : \Delta^0 \to X$ be two objects in a quasicategory X. The **mapping space** $\mathrm{Map}_X(x, y)$ is defined as the pullback

$$\begin{array}{ccc} \mathrm{Map}_X(x,y) & \longrightarrow & \mathrm{Map}_{\mathrm{sSet}}(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \Delta^0 & \xrightarrow{(x,y)} & X \times X \end{array}$$

in simplicial sets.

It is not at all obvious that this is a Kan complex, so that is something we would have to check. See [Lur09, Section 1.2.2].

3.4. **The quasicategory of spaces and the quasicategory of quasicategories.** So far we have only seen examples of quasicategories arising as the nerve of an ordinary category or as the singular complex of a topological space. However, the theory of quasicategories would not be very interesting unless we could produce examples of quasicategories that do not fall under these two classes. Two big players that we have to construct in another manner is the quasicategory of spaces and the quasicategory of quasicategories. These are constructed using another model for ∞-categories, namely simplicial categories.

Definition 3.4. A **simplicial category** is a category enriched in simplicial sets. We denote the category of simplicial categories by Cat_Δ.

Like quasicategories, simplicial categories provide us with a model for ∞-categories. This model might not be as flexible as the quasicategory one, but it has the advantage that it is easy to construct examples of simplicial categories. Every simplicial category is of course a itself a category, so it makes sense to apply the nerve functor to it to obtain a simplicial set. However, this clearly disregards the simplicial structure on our category, so perhaps a refinement of the nerve functor is in order. First, we introduce simplicial category replacements for the sets $[n]$.

Definition 3.5. The simplicial category $\mathfrak{C}[\Delta^n]$ has as objects the numbers 0, 1, …, and n. To construct the mapping complex from i to j: Consider the poset

$$P_{i,j} = \{I \mid \{i,j\} \subseteq I \subseteq [i,j]\}$$

ordered by inclusion, where $[i,j] = \{i, i+1, \ldots, j-1, j\}$. We let

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = \begin{cases} NP_{i,j} & \text{if } i \leq j \\ \emptyset & \text{otherwise}. \end{cases}$$

The composition is induced by the union of subsets.

Note that the construction above provides us with a cosimplicial object $\mathfrak{C}[\Delta^\bullet] : \Delta \to \mathrm{Cat}_\Delta$.

Definition 3.6. The **simplicial nerve** of the simplicial category $\mathscr{C}$ is defined as the simplicial set

$$\mathfrak{N}(\mathscr{C}) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^\bullet], \mathscr{C})$$

The simplicial nerve of a simplicial category is a quasicategory if and only if all mapping complexes in our category are Kan complexes[4]. Let us denote the subcategory of sSet spanned by the Kan complexes by Kan. We regard this a simplicial category and note that all the mapping simplicial sets are indeed Kan complexes.

Definition 3.7. The **quasicategory of spaces** is defined as the simplicial nerve:

$$\mathcal{S} = \mathfrak{N}(\mathrm{Kan}).$$

Remark 3.8. Let CW be the category of CW-complexes and continuous maps between them. It is well-known to homotopy theorists that there is a way to formally invert the homotopy equivalences to obtain the category $\mathcal{H}$, the homotopy category of CW-complexes, whose objects are still CW-complexes, but whose morphisms are now homotopy classes of continuous maps. One important feature of CW-complexes is that they are very flexible, in that we have access to all colimits and limits. This provides us with a multitude of ways to construct new spaces from existing ones. However, it is well-known that colimits and limits do not interact well with homotopy equivalences. For a simple example consider the two diagrams

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^0 & \longrightarrow & I \\ \downarrow & & \\ I & & \end{array} \quad ,$$

where the two points of S^0 are sent to the endpoints of the intervals I in the second diagram. The two diagrams are levelwise equivalent, but note that their respective colimits, the pushouts, are not. The pushout of the left hand side is $*$, but the pushout of the right hand side is the circle S^1, which is certainly not contractible. Classically, the solution is to introduce homotopy colimits, but a priori this is just some method of constructing something that behaves like a colimit and is invariant under levelwise equivalences of diagrams. In particular, they do not have a universal property in the category CW. Nor do they have a universal property in the category $\mathcal{H}$. Indeed, while every diagram in CW certainly has a homotopy colimit, the category $\mathcal{H}$ itself does not admit all colimits. The ∞-category solution is to introduce something lying inbetween CW and $\mathcal{H}$, and this is precisely what we have done above. Indeed, we have a diagram

$$\mathrm{CW} \to \mathcal{S} \to \mathcal{H},$$

of quasicategories where $\mathcal{S}$ has the following crucial properties[5]:

[4] This is part of a larger identification of the homotopy category of simplicial categories with that of quasicategories. In [Ber05], Bergner sets up a model structure on the category of simplicial categories with the so-called Dwyer-Kan equivalences, first introduced in [DK80], as the weak equivalences. In [Lur09, Section 2.2.5] it is shown that this is Quillen equivalent to the category of simplicial sets with the Joyal model structure. A simplicial category is sent to a quasicategory, a fibrant objects of the Joyal model structure, precisely when it is fibrant in the Bergner model structure, which is furthermore equivalent to the statement that all the mapping simplicial sets are Kan complexes.

[5] I want to thank Markus Land for explaining this story to me; his discussion of the homotopy category of spaces vs. the ∞-category was very enlightening.

- The functor $\mathrm{CW} \to \mathcal{S}$ is initial among all functors from $\mathrm{CW} \to \mathscr{C}$ to a quasicategory $\mathscr{C}$ which sends homotopy equivalences to equivalences.
- The homotopy category of $\mathcal{S}$ is $\mathcal{H}$.
- As we will soon see, there are quasicategorical interpretations of colimits and limits, with suitable universal properties. The quasicategory $\mathcal{S}$ has all of these.
- A homotopy (co)limit in CW is mapped to a (co)limit in $\mathcal{S}$.

We can also construct the quasicategory of quasicategories in a similar fashion. Let us construct the simplicial category $\mathrm{Cat}_\infty^\Delta$ whose objects are (small) quasicategories and whose mapping simplicial sets is given by

$$\mathrm{Hom}_{\mathrm{Cat}_\infty^\Delta}(X, Y) = \mathrm{Map}_{\mathrm{sSet}}(X, Y)^{\simeq}\,.$$

Now we just apply the simplicial nerve.

Definition 3.9. The **quasicategory of (small) quasicategories** is the simplicial nerve:

$$\mathrm{Cat}_\infty = \mathfrak{N}(\mathrm{Cat}_\infty^\Delta)\,.$$

Remark 3.10. The quasicategory Cat_∞ has all small colimits and limits. This allows us to quite directly study presheaves $\mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ and to formulate what it means for this to be a sheaf given some Grothendieck topology on $\mathscr{C}$.

4. Categorical notions

We want to be able to generalize basic concepts from ordinary category theory to the context of quasicategories. In this section we go through the basic notions the reader might have encountered in category theory and give corresponding definitions in the model of quasicategories.

4.1. **Functors and natural transformations.**

With quasi-categories as a model for ∞-categories it is easy to describe what we mean by functors and natural transformation of ∞-categories. Indeed, if ∞-categories are interpreted as quasicategories, then functors between them are simply interpreted as maps of simplicial sets.

Definition 4.1. Let X and Y be quasicategories. The **quasicategory of functors** from X to Y is defined as the mapping simplicial set:

$$\mathrm{Fun}(X, Y) = \mathrm{Map}_{\mathrm{sSet}}(X, Y)\,.$$

This is indeed an quasicategory, see [Lur09, Proposition 1.2.7.3]. We will refer to an object in this ∞-category as a **functor** from X to Y and write $F : X \to Y$ in line with ordinary category theory. Similarly, a morphism in the above ∞-category is referred to as a **natural transformation** of functors from X to Y.

4.2. **Equivalences between quasicategories.**

Depending on our point of view there are many equivalent notions of an equivalence between quasicategories. We can start by defining essentially surjective and fully faithful functors, and then simply define an equivalence of quasicategories to be a functor that is both essentially surjective and fully faithful.

Definition 4.2. A functor $f : X \to Y$ between quasicategories is said to be **essentially surjective** if every object y of Y is equivalent to fx for some x in X.

Definition 4.3. A functor $f : X \to Y$ is **fully faithful** if for every pair of objects x, y in X the induced map $\mathrm{Map}_X(x, y) \to \mathrm{Map}_Y(fx, fy)$ is a homotopy equivalence of Kan complexes.

Definition 4.4. A functor $f : X \to Y$ between quasicategories is an equivalence if it is fully faithful and essentially surjective.

Various references on quasicategories tend to use different definitions for equivalences between quasicategories. For example, in [Cis19] the definition given of an equivalence of quasicategories is essentially that of an equivalence in the quasicategory Cat_∞, so that the definition we have given here becomes a (very non-trivial) theorem [Cis19, Theorem 3.9.7].

4.3. **Colimits and limits.** Before we go for all colimits and limits, let us start with some preliminary definitions. In particular, we want to start with the definition of the join of two simplicial sets. The most straight-forward way of doing this is to use so-called augmented simplicial sets. The **augmented simplex category** Δ^+ is category we get from Δ if we give it an initial object. For reasons that will soon be clear to the reader, we denote this extra initial object by $[-1]$. This category has a monoidal structure, the join, given by

$$[m] \star [n] = [m+n+1]$$

with unit being our initial object $[-1]$[6]. The category of **augmented simplicial sets** is the presheaf category

$$\mathrm{sSet}^+ = \mathrm{Fun}(\Delta^+, \mathrm{Set})\,.$$

The above monoidal structure on Δ^+ induces a monoidal structure, which we will also denote by $\star$, on augmented simplicial sets via Day convolution.

Note that an augmented simplicial set can be viewed as a triple (X, Y, ϵ), where X is a simplicial set, Y is a set (the value of our functor on the initial object $[-1]$), and $\epsilon : X \to Y$ is a map of simplicial sets, where the target is interpreted as a constant simplicial set. This provides us with a forgetful functor $i^* : \mathrm{sSet}^+ \to \mathrm{sSet}$. This functor has a right adjoint $i_* : \mathrm{sSet} \to \mathrm{sSet}^+$ which sends a simplicial set X to the trivially augmented simplicial set $(X, *, \epsilon_X)$, where ϵ_X is the obvious map.

Definition 4.5. Let X and Y be simplicial sets. Their **join** is defined as the simplicial set

$$X \star Y = i^*(i_*(X) \star i_*(Y))\,.$$

Explicitly, the n-simplicies of the join is given by

$$(X \star Y)_n = \coprod_{i+j+1=n} X_i \times Y_j\,.$$

where we take X_{-1} and Y_{-1} to be $*$. If both X and Y are quasicategories, then their join is also a quasicategory; see [Lur09, Proposition 1.2.8.3].

Definition 4.6. Let $p : Y \to X$ be a map of simplicial sets. We define

$$X_{/p} = \mathrm{Hom}_p(\Delta^{\bullet} \star Y, X) \quad \text{and} \quad X_{p/} = \mathrm{Hom}_p(Y \star \Delta^{\bullet}, X)$$

where the subscript on the right hand sides indicates that we consider only those morphisms f such that $f\mid_Y = p$.

If X is an ∞-category then the simplicial sets described above are also ∞-categories referred to as the **∞-category of cones** on p and the **∞-category of cocones** on p, respectively. Now let us define arguably the easiest colimits and limits.

Definition 4.7. Let X be a quasicategory.

(1) We say that an object $\emptyset : \Delta^0 \to X$ is **initial** if the mapping space $\mathrm{Map}_X(\emptyset, x)$ is contractible for all objects $x : \Delta^0 \to X$.
(2) We say that an object $* : \Delta^0 \to X$ is **terminal** if the mapping space $\mathrm{Map}_X(x, *)$ is contractible for all objects $x : \Delta^0 \to X$.
(3) We say that an object $0 : \Delta^0 \to X$ is a **zero object** if it is both initial and terminal.

If a quasicategory X has initial/terminal/zero objects, then those initial/terminal/zero objects form a contractible Kan complex [Lur09, Proposition 1.2.12.9], so it makes sense to speak of *the* initial/terminal/zero object, at least from a higher categorical point of view. The same line of thought holds for the following definition.

Definition 4.8. Let $p : K \to X$ be a map of simplicial sets where X is a quasicategory.

[6]The join of two order-preserving maps $f : [m] \to [m']$ and $g : [n] \to [n']$ is given by

$$(f \star g)(i) = \begin{cases} f(i) & \text{if } 0 \le i \le m \\ g(i-(m+1)) + (n+1) & \text{if } m+1 \le i \le m+n+1 \end{cases}.$$

Note in particular that this is indeed a non-symmetric monoidal structure.

(1) The limit of $p : K \to X$ is the final object in $X_{/p}$ (if it exists).
(2) The limit of $p : K \to X$ is the terminal object of $X_{p/}$ (if it exists).

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