K-theory & DAG

Talk 5 - Reduction to K-theory of spectra

Recall: $\mathcal{C}$ ptd. ∞-cat w/ ~~p~~ fin. colim ⇝ Valdhausen construction

$S_\bullet(\mathcal{C}) = \mathrm{Gap}_{[\bullet]}(\mathcal{C})^{\simeq}$ (a simplicial Kan cmplx) ⇝ "$K(\mathcal{C})$"

K-theory space $K(\mathcal{C}) = \Omega|S_\bullet(\mathcal{C})|$ (an infinite loop space)

The Additivity theorem

$\mathcal{C}$ and $\mathcal{D}$ ptd. ∞-cat w/ fin. colim

Thm (Add.): $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}\times\mathcal{C}$, $(X \xrightarrow{f} Y) \mapsto (X, \mathrm{cofib}\, f)$ induces an equiv. $K(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to K(\mathcal{C}\times\mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C})$.

Consider:

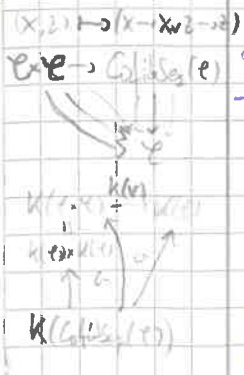
$$\begin{tikzcd}
 & & (X \to X\vee Z \to Z) & \\
(X \xrightarrow{f} Y \xrightarrow{g} Z) \arrow[rrr, mapsto, bend left] \arrow[dddd, mapsto, bend right] & & & \\
\mathrm{CofibSeq}(\mathcal{C}) \arrow[d, "\simeq"] \arrow[rrd] & & & (X, Z) \arrow[uul, mapsto, dashed] \arrow[dddll, mapsto, dashed, bend left] \\
\mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[rrr] \arrow[rr, phantom, "\circlearrowleft"] & & & \mathcal{C}\times\mathcal{C} \arrow[ulll, dashed, bend right, "\text{split}"'] \arrow[lll, dashed, bend left, "\text{split}"] \\
(X \xrightarrow{f} Y) \arrow[rrr, mapsto] & & & (X, \mathrm{cofib}\, f) \\
 & (X \to X\vee Z) & &
\end{tikzcd}$$

~~Cor~~

Cor: All functors in this diagram induce equiv. after $K(-)$.

Cor: (i) Let $G', G, G'' : \mathrm{CofibSeq}(\mathcal{C}) \to \mathcal{C}$ given by sending $X \to Y \to Z$ to X, Y, resp. Z. Then $G_* \simeq G'_* + G''_*$: in $\mathrm{Hom}\{K_\bullet(\mathrm{CofibSeq}(\mathcal{C})) \to K_\bullet(\mathcal{C})\}$.

(ii) More generally: $F' \to F \to F''$ cofib seq. of functors $\mathcal{C} \to \mathcal{D}$ preserving fin. colimits. Then $F_* \simeq F'_* + F''_*$



Pf: (i) By above enough for split sequences: $X \to X\vee Z \to Z$. But for these ~~clearly~~ follows from def. of E_∞-str. + on $K_\bullet$.

(ii) follows from (i) by factoring through $\mathrm{Cofib}(\mathcal{D})$. □

Cor: $\mathrm{id} + \Sigma_* \simeq 0 : K(\mathcal{C}) \to K(\mathcal{C})$, ie. $\Sigma_* =$ "$(-1)\cdot$" in part. Σ_* is equiv. (check on htpy grps

Pf: $\mathrm{id} \to * \to \Sigma$ cofib. seq.

Def: $\mathrm{SW}(\mathcal{C}) := \operatorname{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \to \cdots)$ the Spanier-Whitehead category of $\mathcal{C}$

Fact: $\mathrm{SW}(\mathcal{C})$ is stable.

Cor: The canonical functor induces an equiv. $K(\mathcal{C}) \xrightarrow{\simeq} K(\mathrm{SW}(\mathcal{C}))$.

In part. can always reduce to stable ∞-cat

Pf: Follows from Σ_* equiv. and:

Fact: K preserves filtered colimits

Pf Sketch (Thm): Have fiber seqs:

$0 \mapsto (* \to 0) \qquad (C \to D) \mapsto C$

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, equal] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r] & \mathcal{C} \arrow[d, equal] \\
\mathcal{C} \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{C} \times \mathcal{C} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{C}
\end{tikzcd}$$

$(C \to C \vee D) \mapsfrom (C, D)$

$\xmapsto{K}$ Assume: fib. seq $\Rightarrow$ Thm

$\xmapsto{K}$ fib seq. — Enough to show

$D \mapsto (*, D) \qquad (C, D) \mapsto C$

Easy: fib. seq. after $S_\bullet(-)$

$\rightsquigarrow$ use (technical) criteria for $|-|$ being again a fib. seq. $\square$

Ind-objects & idempotent completion

Idea: $\mathcal{C}$ ∞-cat w/ fin. colimits $\underset{\text{all (small) colimits}}{\overset{\text{formally add}}{\rightsquigarrow}}$ $\mathrm{Ind}(\mathcal{C})$ ∞-cat of Ind-objects in $\mathcal{C}$

More precisely $\mathrm{Ind}(\mathcal{C})$ characterized by:

a) $\exists j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ fully faithful

b) $\mathrm{Ind}(\mathcal{C})$ admits filtered colimits (actually admits all small colimits) & j preserves colimits

c) Every object in $\mathrm{Ind}(\mathcal{C})$ is filtered colimit $\operatorname{colim}_\alpha C_\alpha$ with $C_\alpha \in \mathcal{C}$

d) $C \in \mathcal{C}$ is compact in $\mathrm{Ind}(\mathcal{C})$ ($\Leftrightarrow \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C, -)$ pres. filtered colimits)

Why these conditions?

Let "$\operatorname{colim}_\alpha C_\alpha$" denote "formally added colimit":

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\text{“}\operatorname{colim}_\alpha C_\alpha\text{”}, \text{“}\operatorname{colim}_\beta D_\beta\text{”}) \overset{\text{always (u.p.)}}{\simeq} \lim_\alpha \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, \text{“}\operatorname{colim}_\beta D_\beta\text{”}) \overset{\text{cpt. nes.}}{\simeq}$$

$$\simeq \lim_\alpha \operatorname{colim}_\beta \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C_\alpha, D_\beta) \overset{\text{fully faithful}}{\simeq} \lim_\alpha \operatorname{colim}_\beta \mathrm{Map}_{\mathcal{C}}(C_\alpha, D_\beta)$$

Rem: One construction is: $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ full subcat of functors pres. finite limits.

Ex: $\mathrm{Ind}(\mathrm{Grp}^{\mathrm{fin.pres.}}) \simeq \mathrm{Grp}$ (similarly for Ab, Ring, CRing, ...), $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}) \simeq \mathcal{S}$

R ring spectrum $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{perf}}) \simeq \mathrm{Mod}_R$ (perfect $\Leftrightarrow$ compact)

Q: What are the cpt. $\overset{\mathcal{C}}{\cap}$ obj. of $\mathrm{Ind}(\mathcal{C})$?

Let $\mathrm{colim}_\alpha C_\alpha \in \mathrm{Ind}(\mathcal{C})^{cpt} \Rightarrow$

$$\begin{tikzcd}
\mathrm{colim}_\alpha C_\alpha \arrow[rr, "id"] \arrow[dr] & \circlearrowleft & \mathrm{colim}_\alpha C_\alpha \\
& C_{\alpha_0} \arrow[ur, "can"'] &
\end{tikzcd}$$

$\Rightarrow \mathrm{colim}_\alpha C_\alpha$ retract of C_{α_0}

Fact: $\mathrm{Ind}(\mathcal{C})^{cpt} =$ retracts of objects $C \in \mathcal{C}$.

Def: $\overline{\mathcal{C}} := \mathrm{Ind}(\mathcal{C})^{cpt}$ is the idempotent completion of $\mathcal{C}$. ("formally add $\overset{\text{images of idempotents}}{\text{~~retracts~~}}$")

$\mathcal{C}$ is idempotent complete if $\mathcal{C} \xrightarrow{\simeq} \overline{\mathcal{C}}$ eq.

$$\begin{tikzcd}
X \arrow[rr, "id"] \arrow[dr, "i"'] & & X \arrow[dr, "i"] & \\
& C \arrow[ur, "r"] \arrow[rr, "r"'] & \circlearrowleft & C
\end{tikzcd}$$

Have: $r \circ r \simeq r$

So: retracts $\overset{\text{(weak)}}{\rightsquigarrow}$ idempotent

Q: Other direction?

In this situation: $X \simeq \mathrm{colim}(C \xrightarrow{r} C \xrightarrow{r} C \to \dots)$

$$\begin{tikzcd}
C \arrow[r, "r"] \arrow[d] & C \arrow[r, "r"] \arrow[d] & C \arrow[r] \arrow[d] & \dots \\
X \arrow[r, "id"'] \arrow[ur] & X \arrow[r, "id"'] \arrow[ur] & X \arrow[ur] &
\end{tikzcd}$$

So, let $C \xrightarrow{r} C$ st. $h_r: r \circ r \xrightarrow{\sim} r$. Set $X := \mathrm{colim}(C \xrightarrow{r} C \xrightarrow{r} C \to \dots)$.

Want $X \to C \to X$ with $\overset{id}{\frown}$ retraction

$$\begin{tikzcd}
C \arrow[r, "r"] \arrow[d, "r"'] & C \arrow[r, "r"] \arrow[d, "r"'] & C \arrow[r, "r"] \arrow[d, "r"'] & \dots & \overset{\mathrm{colim}}{\rightsquigarrow} X \arrow[d] \arrow[dd, bend left, "\simeq"] \\
C \arrow[r, "id"] \arrow[d, "r"'] & C \arrow[r, "id"] \arrow[d, "r"'] & C \arrow[r, "id"] \arrow[d, "r"'] & \dots & \overset{\mathrm{colim}}{\rightsquigarrow} C \arrow[d] \\
C \arrow[r, "r"] & C \arrow[r, "r"] & C \arrow[r, "r"] & \dots & \rightsquigarrow X
\end{tikzcd}$$

if

$$\begin{tikzcd}
r \circ r \circ r \arrow[r, "h \circ r"] \arrow[d, "r \circ h"'] & r \circ r \arrow[d, "h"] \\
r \circ r \arrow[r, "h"] \arrow[ur, phantom, "\circlearrowleft"] & r
\end{tikzcd}$$

ie r coherent idempotent

(Enough to get all coherences)

So: $r: C \to C$ coherent idempotent $\rightsquigarrow X = \mathrm{colim}(C \xrightarrow{r} C \xrightarrow{r} \dots) \in \mathrm{Ind}(\mathcal{C})$

retract of C ~~$\Leftrightarrow$~~ $\Rightarrow X \in \overline{\mathcal{C}}$ "formal image of r"

skipped in talk

Warning: $\mathcal{C}$ 1-cat w/ fin. colim $\Rightarrow X \simeq \mathrm{coeq}(r, id) \Rightarrow \mathcal{C} = \overline{\mathcal{C}}$

retract in $\mathcal{S}$ of finite $\Updownarrow$

But wrong for ∞-cat (eg. $\mathcal{S}^{fin}$)) (~~X~~ $X \in \mathcal{S} = \mathrm{Ind}(\mathcal{S}^{fin})$ cpt $\Leftrightarrow$ fin. dom)

Facts: • $\overset{\mathcal{C} \text{ stable} \Downarrow}{\overline{\mathcal{C}} \text{ stable}}$ / all $C' \in \overline{\mathcal{C}}$ are direct summand of $C \in \mathcal{C}$

• $\mathcal{C}$ std. w/ fin. colim $\Rightarrow \mathrm{Sw}(\overline{\mathcal{C}}) \simeq \overline{\mathrm{Sw}(\mathcal{C})}$ hence

skipped in talk

~~Reduction to K-theory of ring spectra~~

~~Step 1 $K(\mathcal{C}) \simeq K(SW(\mathcal{C}))$ ⇝ can assume $\mathcal{C}$ stable~~

~~Step~~ Thm: Let $\mathcal{C}$ ptd ∞-cat w/ fin colim, $\mathcal{C}_0 \subseteq \mathcal{C}$ full sub, closed under fin. colim. Assume ~~[illegible]~~ every $C \in \mathcal{C}$ is a retract of some $C_0 \in \mathcal{C}_0$. Then

$$\begin{tikzcd} K(\mathcal{C}_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K(\mathcal{C}) \arrow[d] \\ K_0(\mathcal{C}_0) \arrow[r] & K_0(\mathcal{C}) \end{tikzcd}$$

htpy pullback. In part. $K_n(\mathcal{C}_0) \to K_n(\mathcal{C})$ iso for $n \geq 1$!

Pf: Again easy simplicially, (more) work for after $|\cdot|$.

Cor: $K_n(\mathcal{C}) \xrightarrow{\simeq} K_n(\overline{\mathcal{C}})$ for $n \geq 1$

Reduction to K-theory of ring spectra

Let $\mathcal{C}$ ptd ∞-cat w/ fin. colim.

Step 1: $K(\mathcal{C}) \simeq K(SW(\mathcal{C}))$ ⇝ can assume $\mathcal{C}$ stable

Step 2: $K_n(\mathcal{C}) \cong K_n(\overline{\mathcal{C}})$ for $n \geq 1$ ⇝ assume $\mathcal{C}$ idemp. compl. (lose K_0)

Step 3:

Def: ~~[illegible]~~ $\mathcal{C}$ stable ∞-cats. $C \in \mathcal{C}$ is a generator if $C \in \mathcal{C}_0 \subseteq \mathcal{C}$ w/ $\mathcal{C}_0$ stable and idemp. compl. implies $\mathcal{C}_0 = \mathcal{C}$. ($\mathcal{C}$ is smaller [illegible])

Note: $\mathcal{C}$ is the filtered union (and thus colimit) of its subcats ~~with~~ that have a generators

(filtered follows from $\mathcal{C}\langle A\rangle \subseteq \mathcal{C}\langle A \oplus B\rangle \supseteq \mathcal{C}\langle B\rangle$ also)

⇝ So, $K(\mathcal{C}) \simeq K(\operatorname{colim} \mathcal{C}\langle G\rangle) \simeq \operatorname{colim} K(\mathcal{C}\langle G\rangle)$ ⇝ assume $\mathcal{C}$ has generator

Step 4: $C \in \mathcal{C}$ ⇝ $\mathrm{End}(\mathcal{C})$ ring spectrum of endomorphisms

(Schwed, Shipley) Thm: There exists $\mathrm{Mod}^{\mathrm{perf}}_{\mathrm{End}(C)} \hookrightarrow \mathcal{C}$ fully faithful. $(\mathrm{Map}_{\mathcal{C}}(C,X) \longleftarrow\!\shortmid X)$

$M \longmapsto$ "$M \wedge_{\mathrm{End}(C)} C$"

It is an equiv $\Leftrightarrow$ C generator.

Hence, Thm 4 in that case: $K(\mathcal{C}) \simeq K(\mathrm{Mod}^{\mathrm{perf}}_{\mathrm{End}(C)}) =: K(\mathrm{End}(C))$