HIGHER TOPOS THEORY

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Abstract. These are corrected and expanded notes of my talk at the *K-theory and Derived Algebraic Geometry* seminar.

We present thee viewpoints on ∞-topoi, following Lurie: via Giraud-type axioms; as left-exact localization functors on presheaf ∞-categories; via sheaves on a ∞-site. As in the classical case, the first two turn out to be equivalent, while the third is not: although sheaves on an ∞-site always form an ∞-topos, not all ∞-topoi arise in thise way. We close by remarking that sheaves and stack really are the same thing in the ∞-setting.

1. ∞-Topoi

Throughout, $\mathcal{S}$ is the ∞-category of spaces.

1.1. **Accessible & Presentable ∞-categories.**

An accessible ∞-category is a category which is 'not too big'. More precisely

Definition 1.1.1. Let κ be a regular cardinal.

- A simplicial set K is *κ-small* if the set of nondegenerate simplices of K has cardinality less than κ;
- An ∞-category $\mathcal{A}$ is *κ-filtered* if for every $K \to \mathcal{A}$ where K is κ-small extends to a map $K^{\triangleright} \to \mathcal{A}$;
- An object x in an ∞-category $\mathcal{C}$ is κ-compact if $\mathcal{C}(x, -)$ commutes with κ-filtered colimits.

Remark 1.1.2. For $\kappa = \omega$ and $\mathcal{A}$ a category, one recovers the classical notion of $\mathcal{A}$ being filtered.

Definition 1.1.3. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is

- *Accessible* if it is generated by a set of κ-compact objects under small colimits, for a regular cardinal κ;
- *Presentable* if it is cocomplete and accessible.

Example 1.1.4 ([HTT, Prop. A.3.7.6])**.** An ∞-category $\mathcal{C}$ is presentable iff it is equivalent to the ∞-category of a combinatorial simplicial model category. Here, combinatorial means presentable (1-categorically) and cofibrantly generated.

For example, $\mathcal{S}$ is presentable. Indeed, finite simplicial sets are ω-compact, and they generate $\mathcal{S}$ under colimits.

Remark 1.1.5 ([HTT, Cor. 5.5.2.4])**.** A presentable ∞-category is also complete.

1.2. **Groupoid objects.**

Let $\mathcal{C}$ be an ∞-category with fiber products.

Definition 1.2.1. Let $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be given. We say that:

(1) *X satisfies the Segal condition* if for all $n \geq 1$ the map $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an equivalence, where the maps are induced by the spine inclusion;
(2) *Morphisms are invertible* in X if the map $X_2 \to X_1 \times_{X_0} X_1$ induced by $\Lambda^2[2] \to \Delta[2]$ is an equivalence.

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A *groupoid object* in $\mathcal{C}$ is an $X \in \mathcal{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ which satisfies the Segal condition and such that morphisms are invertible.

Example 1.2.2. Any classical groupoid G is a groupoid object in $\mathcal{S}\mathrm{et}$. Indeed, (1) says that G is the nerve of a category, and (2) says that all morphisms are isomorphisms.

1.3. **Giraud axioms.**

Definition 1.3.1. Let $f : A \to B$ be a morphism in an ∞-category $\mathcal{C}$ with fiber products.

- The *Čech nerve* $\check{C}(A/B)$ of f is the simplicial diagram $[n] \mapsto A \times_B \cdots \times_B A$ in $\mathcal{C}$.
- f is an *effective epimorphism* if the canonical map $\check{C}(A/B) \to B$ is colimiting.

Definition 1.3.2. An ∞*-topos* is a presentable ∞-category $\mathcal{X}$ such that:

- *Colimits are universal*, meaning that colimits and pullbacks commute in $\mathcal{X}$;
- *Coproducts are disjoint*, meaning that $0 \simeq X \times_{X \sqcup Y} Y$ for all $X, Y \in \mathcal{X}$;
- *Every groupoid object is effective*, meaning that for every groupoid object $G : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{X}$ the natural map $G \to \check{C}(G_0/\operatorname{colim} G)$ is an equivalence.

Remark 1.3.3 ([NSS15, Thm. 2.10])**.** Let $\mathcal{X}$ be an ∞-topos. Then the map $\mathcal{X}_{\mathrm{eff}}^{\Delta[1]} \to \mathcal{G}\mathrm{rpd}(\mathcal{X})$ from the ∞-category of effective epimorphisms in $\mathcal{X}$ to the ∞-category of groupoid objects in $\mathcal{X}$, induced by taking the Čech nerve, is an equivalence.

Example 1.3.4. Let X be a topological space. Then the category $\mathcal{S}\mathrm{h}(X)$ of sheaves on X is a topos, hence an ∞-topos. A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathcal{X}$ is an effective epimorphism iff we can lift sections of $\mathcal{G}$ locally to $\mathcal{F}$, i.e. iff for all $\underline{h}_U \to \mathcal{G}$ there is some $V \subset U$ and $\underline{h}_V \to \mathcal{F}$ such that the following diagram commutes

$$\begin{array}{ccc} \underline{h}_V & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \underline{h}_U & \longrightarrow & \mathcal{G} \end{array}$$

Here, $\underline{h}_{(-)}$ is the Yoneda embedding. So effective epimorphisms are surjections of sheaves.

1.4. **Left-exact localization functors.** Classically, the category $\mathcal{S}\mathrm{h}(X)$ of sheaves on a topological space X comes with an adjunction

$$L : \mathcal{P}(X) \rightleftarrows \mathcal{S}\mathrm{h}(X) : i$$

Here, i is the inclusion into the category $\mathcal{P}(X)$ of presheaves on X, and L is a left-exact left adjoint. This L is of course sheafification, and is called a *localization*.

More generally, any topos comes about as a left-exact localization of the category of presheaves on a category. This is also true in the ∞-setting.

Theorem 1.4.1 ([HTT, Thm. 6.1.0.6])**.** *An ∞-category $\mathcal{X}$ is an ∞-topos iff $\mathcal{X}$ is accessible and there is an adjunction*

$$L : \mathcal{P}_\infty(\mathcal{C}) \rightleftarrows \mathcal{X} : i$$

with L left-exact and i fully faithful, where $\mathcal{P}_\infty(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ for some small ∞-category $\mathcal{C}$.

The functor L is called a *left-exact localization* of $\mathcal{P}_\infty(\mathcal{C})$.

Sketch of one direction. Let $\mathcal{X}$ be an ∞-topos. Then from $\mathcal{X}$ being presentable, Lurie constructs a small, finitely complete ∞-category $\mathcal{C}$ consisting of κ-compact objects in $\mathcal{X}$, such that the inclusion $\mathcal{C} \to \mathcal{X}$ induces an equivalence $\mathrm{Ind}_\kappa(\mathcal{C}) \simeq \mathcal{X}$.

By an adjoint functor argument, the inclusion $i : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{P}_\infty(\mathcal{C})$ has a left adjoint L. One needs to show that L is left exact.

The key lemma in this argument is the following: a functor $\mathcal{P}_\infty(\mathcal{C}) \to \mathcal{X}$ is left-exact if the composition $\mathcal{C} \to \mathcal{X}$ with Yoneda is left-exact. This in fact holds for any small, finitely complete ∞-category $\mathcal{C}$. Now in our case, $\mathcal{C} \to \mathcal{X}$ can be identified with Yoneda $\mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$, which preserves all limits, so is certainly left-exact. □

Example 1.4.2. The ∞-category $\mathcal{S}$ is an ∞-topos. Indeed, we have a left-exact localization $L : \mathcal{P}_\infty(\{\mathrm{pt}\}) \rightleftarrows \mathcal{S} : i$, namely $i = L = \mathrm{id}_\mathcal{S}$.

2. ∞-Sites

2.1. **Grothendieck topologies.**

Definition 2.1.1. Let $\mathcal{T}$ be an ∞-category.

- A *sieve* on $X \in \mathcal{T}$ is a full subcategory $T \subset \mathcal{T}_{/X}$ such that for all $Y' \to Y \in \mathcal{T}_{/X}$ with $Y \in T$ it holds $Y' \in T$;
- For $f : X \to Y$ and a sieve T on Y, we let the *pullback* f^*T be the full subcategory of $\mathcal{T}_{/X}$ spanned by those $U \to X$ such that $U \to X \to Y$ is in T;
- A *Grothendieck topology* τ on $\mathcal{T}$ is a collection of sieves T for each $X \in \mathcal{T}$, called *covering sieves*, such that
  - The trivial sieve $\mathcal{C}_{/X}$ is covering;
  - For $f : X \to Y$ and a covering sieve T on Y, the pullnack f^*T is covering;
  - If T is a covering sieve on X and S any sieve on X such that for any $f \in T$ the pullback f^*S is covering, then S is covering.

An ∞-*site* is an ∞-category endowed with a topology.

Remark 2.1.2. Specifying a topology on an ∞-category $\mathcal{T}$ is equivalent to specifying such on the homotopy category of $\mathcal{T}$.

We are ultimately interested in sheaves with respect to a given Grothendieck topology. As in the classical case, we might have a basis for a given topology, such that the sheaf condition can be tested on a base. More precisely, we introduce the following.

Definition 2.1.3. A *pretopology* $\tilde{\tau}$ on a category $\mathcal{T}$ with pullbacks is an assignment $\{U_\alpha \to X\}_\alpha$ of *coverings*, for each $X \in \mathcal{T}$, such that

- (*Isomorphisms cover*) $\{Y \xrightarrow{\simeq} X\}$ is a covering for each isomorphism $Y \xrightarrow{\simeq} X$;
- (*Transitivity*) If $\{U_\alpha \to X\}_\alpha$ is a covering and $\{V_{\alpha\beta} \to U_\alpha\}_\beta$ is a covering for each α, then $\{V_{\alpha\beta} \to X\}_{\alpha,\beta}$ is a covering;
- (*Stability*) If $\{U_\alpha \to X\}_\alpha$ is a covering, then so is $\{U_\alpha \times Y \to Y\}_\alpha$ for each $Y \to X$;

Let $\mathcal{T}$ be endowed with a pretopology $\tilde{\tau}$. Then the topology *generated by* τ is the topology such that a sieve T on an object X is covering if it contains a cover $\{U_\alpha \to X\}_\alpha$ from $\tilde{\tau}$. We will see that sheaves for τ can be expressed in terms of $\tilde{\tau}$.

Example 2.1.4. Recall that we defined s$\mathcal{R}$ing as the category of product-preserving functors $\mathcal{P}\mathrm{oly}^{\mathrm{op}} \to \mathcal{S}$. Equivalently, s$\mathcal{R}$ing is the ∞-category associated to the model category of simplicial commutative rings.

There are many interesting topologies on s$\mathcal{R}\mathrm{ing}^{\mathrm{op}}$. We will see two of them later on: Zariski and étale. These are generalizations of the classical topologies on $\mathcal{R}\mathrm{ing}^{\mathrm{op}}$, in the sense that a Zariski or étale cover $\{U_\alpha \to X\}_\alpha$ in $s\mathcal{R}\mathrm{ing}^{\mathrm{op}}$ induces a Zariski or étale cover on the underlying discrete rings.

2.2. **Sheaves on an ∞-site.**

Definition 2.2.1. Let $\mathcal{T}$ be an ∞-site.

- For $X \in \mathcal{T}$ and $U \to \underline{h}_X$ a map of presheaves, let $T(U)$ be the full subcategory of $\mathcal{T}_{/X}$ of those $Y \to X$ for which the corresponding morphism $\underline{h}_Y \to \underline{h}_X$ factors through U. Then $T(U)$ is a sieve;
- A presheaf $\mathcal{F} \in \mathcal{P}_\infty(\mathcal{T})$ is a *sheaf* if the map

$$\mathcal{P}_\infty(\mathcal{T})(\underline{h}_X, \mathcal{F}) \to \mathcal{P}_\infty(\mathcal{T})(U, \mathcal{F})$$

is an equivalence, for any $U \to \underline{h}_X$ in $\mathcal{P}_\infty(\mathcal{T})$ such that $T(U)$ is covering.

We let $\mathcal{S}\mathrm{h}_\infty(\mathcal{T})$ be the full sub-∞-category of $\mathcal{P}_\infty(\mathcal{T})$ spanned by the shaves on $\mathcal{T}$.

Proposition 2.2.2. *Let $\tilde{\tau}$ be a pretopology generating the topology τ on an ∞-category $\mathcal{T}$ with fiber products. Then a presheaf $\mathcal{F}$ on $\mathcal{T}$ is a sheaf precisely if for any cover $\{U_\alpha \to X\}_\alpha$ in $\tilde{\tau}$ the map*

$$\mathcal{F}(X) \to \lim_{n\in\mathbf{\Delta}} \left(\prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha \times_X U_\beta) \cdots \right)$$

is an equivalence.

Proof. Let $\{U_\alpha \to X\}_\alpha$ be a cover. Let $\mathcal{U}$ be the colimit of the check nerve of $\coprod U_\alpha \to X$ in $\mathcal{P}_\infty(\mathcal{T})$. Then $\mathcal{U} \to X$ induces a sieve on X as in [HTT, Prop. 6.2.2.5]. The collection of such sieves generates what is called a quasi-topology in [Hoy13, §C]: a collection of sieves stable under pullbacks. The statement follows from [Hoy13, Cor. C.2], since the limit on the right is exactly $\mathcal{P}_\infty(\mathcal{T})(\mathcal{U}, \mathcal{F})$. ☐

Remark 2.2.3. One can thus say that a sheaf $\mathcal{F}$ 'sees covers as effective epimorphism', since it treats $\sqcup U_\alpha \to X$ as effective epimorphism.

If we in stead of space-valued presheaves on $\mathcal{T}$ consider $\mathcal{C}$-valued presheaves on $\mathcal{T}$, we get the notion of $\mathcal{C}$-valued sheaves in the obvious way. We write the resulting category as $\mathcal{S}\mathrm{h}_\mathcal{C}(\mathcal{T})$.

Remark 2.2.4. Let $\mathcal{T}$ be a site with fiber products, $\mathcal{F}$ a presheaf on $\mathcal{T}$. Then $\mathcal{F}$ is a sheaf in the sense of Def. 2.2.1 iff it is so in the classical sense. This follows from Prop. 2.2.2, together with the fact that the inclusion from $[0] \rightrightarrows [1]$ to $\mathbf{\Delta}$ is final.

Note, however, that $\mathcal{S}\mathrm{h}_\infty(\mathcal{T})$ is not the same as $\mathcal{S}\mathrm{h}(\mathcal{T})$, since $\mathcal{S}$-valued sheaves are in general not $\mathcal{S}\mathrm{et}$-valued, even if $\mathcal{T}$ is a 1-category.

Proposition 2.2.5. *Let $\mathcal{T}$ be an ∞-site. Then $\mathcal{S}\mathrm{h}_\infty(\mathcal{T})$ is a topos.*

Example 2.2.6 ([HTT, Prop. 6.5.2.14])**.** Let $\mathcal{T}$ be a site, and let $\mathbb{A}$ be the simplicial model category of simplicial presheaves on $\mathcal{T}$, with the local model structure. Roughly, the cofibrations are taken pointwise, and the weak equivalences $\mathcal{F} \to \mathcal{G}$ are those that induce isomorphisms $\pi_n\mathcal{F} \to \pi_n\mathcal{G}$ on homotopy sheaves.

The ∞-category associated to $\mathbb{A}$ is equivalent to the *hypercompletion* of $\mathcal{S}\mathrm{h}_\infty(\mathcal{T})$, which roughly means that we force maps which are isomorphisms on the homotopy sheavs to be equivalences. This gives a link to the topos theory presented in [HAG1].

This remark suggests that not all ∞-topoi are hypercomplete, and that not all ∞-topoi come about as sheaves on an ∞-site. Both remarks are true. See also [MO].

2.3. **Stacks on an ∞-topos.** Let $\mathcal{X}$ be an ∞-topos.

Definition 2.3.1. A *stack* on $\mathcal{X}$ is a limit-preserving functor $\mathcal{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$.

More generally, if $\mathcal{C}$ is any ∞-category, then a $\mathcal{C}$*-valued stack* on $\mathcal{X}$ is a continuous functor $\mathcal{F} : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$.

We write $\mathcal{S}\mathrm{t}_{\mathcal{C}}(\mathcal{X})$ for the category of $\mathcal{C}$-valued stacks on $\mathcal{X}$, and just $\mathcal{S}\mathrm{t}(\mathcal{X})$ in the case $\mathcal{C} = \mathcal{S}$.

Stack and sheaves are basically the same thing, as the following shows.

Proposition 2.3.2 ([SAG, Prop. 1.3.1.7])**.** *Let $\mathcal{T}$ be an ∞-site, $\mathcal{C}$ a complete ∞-category. Then the map $\mathcal{S}\mathrm{t}_{\mathcal{C}}(\mathcal{S}\mathrm{h}_\infty(\mathcal{T})) \to \mathcal{S}\mathrm{h}_{\mathcal{C}}(\mathcal{T})$ induced by composing with Yoneda and sheafification $\mathcal{T} \to \mathcal{P}(\mathcal{T}) \to \mathcal{S}\mathrm{h}(\mathcal{T})$ is an equivalence.*

Remark 2.3.3. Classically, a prestack can be thought of as a category fiberered in groupoids over a base category $\mathcal{C}$, or a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathcal{G}\mathrm{rpd}$. This is also available in the ∞-setting, but now the difference between functors and pseudofunctors disappears. More precisely, we have an equivalence between the ∞-category of right fibrations over a base ∞-category $\mathcal{C}$ and the ∞-category of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. See [HTT, §2.2] for precise statements.

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