# $K$-THEORY AND DAG

## 1. $K$-THEORY

### 1.1. $\infty$-categories.

(1) Reminder on simplicial stuff
(2) Definition of $\infty$-categories (quasicategories)
(3) Functors
(4) Adjunctions
(5) Mapping spaces
(6) $\infty$-category of spaces
(7) Yoneda
(8) Homotopy category
(9) Equivalences in $\infty$-categories
(10) Equivalences between $\infty$-categories
(11) Maximal subgroupoid
(12) Geometric realization
(13) Limit constructions

### 1.2. Stable $\infty$-categories.

(1) Definition of stable $\infty$-categories [Lur09, Lur16a]
(2) Triangulated structure on the homotopy category
(3) Stabilization of $\infty$-categories
(4) Example: derived category of abelian category
(5) $K_0$ of a stable $\infty$-category

### 1.3. Spectra. See also [Gep19], [Lur16a], [YD19]

(1) Definition of spectra and ring spectra
(2) Stabilization
(3) Mapping spectra in stable $\infty$-categories
(4) Compact objects
(5) $\mathrm{Mod}_R, \mathrm{Mod}_R^{\mathrm{perf}}, \mathrm{Mod}_R^{\mathrm{proj}}$ for ring spectra $R$
(6) Connection to (simplicial) commutative rings

### 1.4. $S_\bullet$-construction and $K_0$.

(1) Waldhausen $S_\bullet$ construction on pointed $\infty$-category [Lur14, §14]
(2) Example: $K$-theory of ring spectra via projective modules and via perfect modules [Lur14, §19]
(3) Relation to $K_0$; $K_0$ by hand
(4) Relation to $BGL(A)^+$
(5) Example: $K_0$(finite pointed spaces)

### 1.5. Reduction to $K$-theory of spectra. See also [BGT10]

(1) Additivity theorem [Lur14, §17]
(2) Invariance under stabilization
(3) Ind-objects and idempotent completion [Lur14, §15], [Lur06]
(4) $K(\mathrm{Ind}(\mathcal{C}))$ for stable $\infty$-categories $\mathcal{C}$

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(5) $K$-theory of pointed $\infty$-categories with finite colimits in terms of $K$-theory of ring spectra $\mathrm{End}(C)$ [Lur14, §19]

## 1.6. Additive and Localizing invariants.

(1) (Split) exact sequences of stable $\infty$-categories
(2) Additive invariants [BGT10]
(3) Universal property of connective $K$-theory
(4) The localization theorem for $K$-theory

## 1.7. Non-connective $K$-theory.

(1) Localizing invariants
(2) Non-connective $K$-theory
(3) Relation between connective and non-connective $K$-theory
(4) Universal property of non-connective $K$-theory
(5) Examples: for instance $K$-theory of open immersion of schemes
(6) Weibel's conjecture

## 1.8. Connections to classical $K$-theory.

(1) Exact 1-categories (Potentially restric to abelian categories)
(2) $K$-theory of exact 1-categories
(3) Theorem of the heart
(4) Classical results: Devissage, the localization sequence for $\mathbb{Z} \to \mathbb{Q}$

## 2. DAG

## 2.1. Higher topos theory.

(1) Grothendieck topologies
(2) $\infty$-sites
(3) Left-exact localization functors
(4) Groupoid objects
(5) $\infty$-topoi
(6) Example of $\infty$-topoi: $\infty$-sheaves on $\infty$-sites
(7) $\infty$-stacks on $\mathrm{Sh}(T)$ are equivalently $\infty$-sheaves on $T$, for $\infty$-topoi $T$

## 2.2. Derived schemes. We can follow [Kha16], [Mat12]

(1) Derived rings, the associated $E_\infty$-rings, the underlying discrete rings
(2) Open immersions of derived affine schemes
(3) Zariski topology on s$\mathcal{R}$ing
(4) Open immersions of Zariski stacks
(5) Derived schemes

## 2.3. Derived Stacks. We can follow [Kha16], [Mat12]

(1) Cotangent complex
(2) Étale topology
(3) Derived étale stacks
(4) Quasi-coherent modules on stacks: connective and non-connective, the case of derived schemes
(5) Smooth morphisms of derived schemes
(6) Derived Artin stacks

## 2.4. Perfect complexes and $K$-theory of stacks.

(1) Quasi-coherent sheaves on stacks, Artin stacks, schemes
(2) The adjunction $f^* \dashv f_*$ for a map of stacks $f : \mathcal{X} \to \mathcal{Y}$
(3) Perfect complexes on derived stacks [BZFN10], [Lur16b], Khan lecture notes
(4) Symmetric monoidal $\infty$-categories: dualizable objects
(5) Compact objects
(6) The relations between perfect complexes, dualizable object and compact object in $\mathrm{QCoh}(X)$
(7) $f^*$ preserves perfect complexes
(8) $\mathcal{P}\mathrm{erf}(X)$ is stable $\infty$-cat

## 2.5. $K$-theory of projective bundles.

(1) Locally free modules of finite rank
(2) Projective bundles on stacks and Serre's twisting sheaf [Kha18b, §6]
(3) Semi-orthogonal decompositions of stable $\infty$-categories [Kha18b], [YD19]
(4) Additive invariants 'split' semi-orthogonal decompositions [Kha18b, Lem. 2.3.3]
(5) Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$

## 2.6. $K$-theory of blow-ups.

(1) Quasi-smooth immersions of derived Artin stacks
(2) Derived blow-ups in quasi-smooth centers
(3) Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$

## 2.7. Zariski Descent. See [Kha18b, §3,4]

(1) Quasi-compact open immersions of derived schemes
(2) qcqs derived schemes
(3) Perfect complexes that vanish on open subsets
(4) Zariski descent for squares
(5) Zariski descent for Čech covers

## 2.8. Grothendieck-Riemann-Roch I. See [Kha18a, §9]

(1) Derived Sym and $\Lambda$
(2) Additive $K$-theory (earlier?)
(3) $\lambda$-rings, $\gamma$-operations, associated grading
(4) $\lambda$-ring structure on $K_0(R) = K_0^{\oplus}(\mathcal{M}\mathrm{od}_R^{\mathrm{proj}})$
(5) Global resolution property
(6) Globalization of $\lambda$-ring structure to $K_0(X)$ for derived scheme $X$

## 2.9. Grothendieck-Riemann-Roch II. See [Kha18a, §10]

(1) Formulation of GRR-theorem for derived schemes

## References

[BGT10]  Andrew J. Blumberg, David Gepner, and Goncalo Tabuada. A universal characterization of higher algebraic k-theory, 2010.
[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and drinfeld centers in derived algebraic geometry. *Journal of the American Mathematical Society*, 23(4):909–966, 2010.
[Gep19]  David Gepner. An introduction to higher categorical algebra. *arXiv preprint arXiv:1907.02904*, 2019.
[Kha16]  Adeel Khan. *Motivic homotopy theory in derived algebraic geometry*. PhD thesis, 2016.
[Kha18a] Adeel Khan. The Grothendieck–Riemann–Roch theorem . *Lecture notes*, 2018.
[Kha18b] Adeel A. Khan. Descent by quasi-smooth blow-ups in algebraic k-theory, 2018.
[Lur06]  J. Lurie. *Higher Topos Theory*. Princeton University Press, August 2006.

[Lur09]    Jacob Lurie. Derived algebraic geometry I: stable $\infty$-categories. *Preprint. http://arxiv. org/abs/math/0608228*, 2009.

[Lur14]    Jacob Lurie. Algebraic K-Theory and Manifold Topology. *Lecture notes*, 2014.

[Lur16a]   Jacob Lurie. Higher algebra. 2014. *Preprint, available at http://www. math. harvard. edu/˜ lurie*, 2016.

[Lur16b]   Jacob Lurie. Spectral algebraic geometry. *Preprint, available at www. math. harvard. edu/˜ lurie/papers/SAG-rootfile. pdf*, 2016.

[Mat12]    Akhil Mathew. Simplicial commutative rings, 2012.

[YD19]     Lucy Yang and Jack Davies. Algebraic K-theory - European Talbot 2019 notes . *Lecture notes*, 2019.