Universality and localization for K-theory

1 Introduction

This document consists of expanded lecture notes for a talk given in the K-theory and Derived algebraic geometry seminar in Stockholm. I will discuss two important properties of algebraic K-theory of stable ∞-categories. The first is its universal property, which I will only describe and not prove. The second is the behavior of algebraic K-theory on certain "exact" sequences of stable ∞-categories, which is described by the Localization Theorem which appears as 5.8 below. In the literature versions of this result are often stated for idempotent complete categories. I wanted to make the point that idemopotent completion is unnecessary (though certainly desirable in many applications), so it does not appear at all in the section on the Localization Theorem. The proof of the localization theorem is an adaptation of Waldhausen's proof of the (Generic) Fibration Theorem [Wal85, Theorem 1.6.4] to this specific setting.

This document is somewhat informal and has not been thoroughly checked for mistakes. Comments are welcome!

2 K-theory as a spectrum and as a space

Recall that for a pointed ∞-category $\mathcal{C}$ with pushouts we define the algebraic K-theory space by

$$K(\mathcal{C}) := \Omega|S_\bullet\mathcal{C}|.$$

The coproduct on $\mathcal{C}$ induces an E_∞-monoid structure on the space $|S_\bullet\mathcal{C}|$ and hence also on its loop space $K(\mathcal{C})$. The E_∞-structure induces a commutative monoid structure on $\pi_0 K(\mathcal{C})$ which agrees with the loop product, by the Eckmann-Hilton argument. This monoid structure is therefore a group structure. Recall that a homotopy-associative monoid structure, such as an E_∞-structure, on a space X is called grouplike if the induced monoid structure on $\pi_0(X)$ is a group. It follows from the above that $K(\mathcal{C})$ is a grouplike E_∞ space. There is adjunction

$$B^\infty : Mon_{E_\infty}(\mathcal{S}) \leftrightarrows Sp : \Omega^\infty,$$

which restricts to an equivalence of categories

$$B^\infty : Grp_{E_\infty}(\mathcal{S}) \leftrightarrows Sp_{\geq 0} : \Omega^\infty,$$

between the full subcategories of grouplike E_∞-spaces on the left hand side and of connective spectra on the right hand side. Using these equivalences allows to view $K(\mathcal{C})$ either as a grouplike E_∞-space or as a connective spectrum.

3 Background on stable ∞-categories

We write Cat_∞^{ex} for the ∞-category of small stable ∞-categories and exact functors between them.

Definition 3.1. A sequence of maps $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in Cat_∞^{ex} is said to be *strict-exact* if $g \circ f \simeq 0$ and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & \mathcal{C} \end{array}$$

is both cartesian and cocartesian in Cat_∞^{ex}. The sequence is called a *split-exact* if it is strict-exact and the functor g admits a right adjoint r such that the counit of the adjunction $\epsilon: g \circ r \to Id_{\mathcal{C}}$ is an equivalence.

Remark 3.2. a) Note that $g \circ f \simeq 0$ is a condition and not a property, so being a strict-exact sequence is a property for sequences in Cat_∞^{ex}.

b) When g has a right adjoint r, the counit $\epsilon: g \circ r \to Id_{\mathcal{C}}$ is an equivalence if and only if r is fully faithful.

Example 3.3. The ring map $f: \mathbb{Z} \to \mathbb{Q}$ induces an exact functor $\mathbb{Q} \otimes_{\mathbb{Z}} (-): Perf(\mathbb{Z}) \to Perf(\mathbb{Q})$. The "kernel" of this functor is the full subcategory $Perf_{tors}(\mathbb{Z})$ of perfect complexes whose homology is torsion. The resulting sequence

$$Perf_{tors}(\mathbb{Z}) \to Perf(\mathbb{Z}) \to Perf(\mathbb{Q})$$

is a strict-exact sequence which is not split.

It will be useful to reformulate the additivity theorem as follows:

Theorem 3.4. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ be a split-exact sequence, and let r be a right adjoint to g. Then the functor*

$$\mathcal{A} \times \mathcal{C} \xrightarrow{f \oplus r} \mathcal{B}$$

induces an equivalence of spaces

$$K(\mathcal{A}) \times K(\mathcal{C}) \xrightarrow{\simeq} K(\mathcal{A} \times \mathcal{C}) \xrightarrow{\simeq} K(\mathcal{B}).$$

Proof. Let $k: \mathcal{B} \to \mathcal{B}$ be the exact functor defined by $k(X) = fib(\eta_X: X \to rg(X))$. Since g takes η_X to an equivalence it follows that k lands in the essential image of f. Since f is fully faithful we may interpret k as the composite $f \circ s$ where $s: \mathcal{B} \to \mathcal{A}$ is an (essentially uniquely determined) exact functor and $s \circ f \simeq Id_{\mathcal{A}}$. Now consider the functor $\mathcal{B} \xrightarrow{(s,g)} \mathcal{A} \times \mathcal{C}$. The composite $(s, g) \circ f \oplus r: \mathcal{A} \times \mathcal{C} \to \mathcal{A} \times \mathcal{C}$ is equivalent to the identity. The other composite is equivalent to $k \oplus (r \circ g)$, which is not an equivalence. However, by the Additivity Theorem from last talk the cofiber sequence of functors

$$k \to Id_{\mathcal{B}} \to r \circ g,$$

induces the middle homotopy in the sequence

$$id_{K(\mathcal{B})} \simeq (Id_{\mathcal{B}})_* \simeq k_* + (r \circ g)_* \simeq (k \oplus (r \circ g))_*.$$

It follows that $(f \oplus r)_*$ and $(s, g)_*$ are mutually inverse equivalences. ☐

Let $\mathcal{C}$ be a small ∞-category W be a set of morphisms in $\mathcal{C}$. A localization of $\mathcal{C}$ by W is an ∞-category $W^{-1}\mathcal{C}$ and a functor

$$\gamma: \mathcal{C} \to W^{-1}\mathcal{C},$$

which sends every morphism in W to an equivalence and such that for any ∞-category $\mathcal{D}$ the functor

$$\gamma^*: Fun(W^{-1}\mathcal{C}, \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$$

induced by γ is fully faithful with image those functors $F: \mathcal{C} \to \mathcal{D}$ which send arrows in W to equivalences. Such localizations always exist, but are difficult to understand in general.

Here is an alternative characterization of strict-exact sequences, which I believe is well-known to experts. Below, we will need condition iii) in the proof of the Localization Theorem.

Lemma 3.5. *A sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in Cat_∞^{ex} with $g \circ f \simeq 0$ is a strict-exact sequence if and only if the following three conditions hold:*

i) The functor f is fully faithful.

ii) The essential image of f is closed under retracts in $\mathcal{B}$.

iii) The functor $g: \mathcal{B} \to \mathcal{C}$ is a localization of $\mathcal{B}$ with respect to the set of maps whose cofiber is in $\mathcal{A}$.

Sketch of proof. We show one implication, which is the one we need. Assume that the sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ is strict-exact, so that the corresponding square is cartesian and cocartesian. Then conditions i) and ii) follow from the cartesian property and condition iii) follows from [NS18, Theorem I.3.3] and the cocartesian property.

□

Remark 3.6. It follows from condition iii) above that the right hand map in a strict-exact sequence induces a Verdier quotient functor on homotopy categories.

4 Universal property of connective K-theory

We will now give a short summary of the universal characterization of algebraic K-theory from the paper [BGT13] of Blumberg-Gepner-Tabuada. Let Cat_∞^{perf} be the full subcategory of Cat_∞^{ex} spanned by the idempotent complete small stable ∞-categories. The functor $Idem(-) := (Ind(-))^\omega$ is a left adjoint to the fully faithful inclusion functor $i: Cat_\infty^{perf} \to Cat_\infty^{ex}$. To formulate the universal property we will restrict the functor K to Cat_∞^{perf}.

Definition 4.1. Let $\mathcal{E}$ be a presentable stable ∞-category. An $\mathcal{E}$-valued *additive invariant* on Cat_∞^{perf} is a functor $E: Cat_\infty^{perf} \to \mathcal{E}$ such that

i) E commutes with filtered colimits.

ii) If $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ is a split exact sequence, and r is a right adjoint to g then the map

$$E(f) + E(r): E(\mathcal{A}) \oplus E(\mathcal{C}) \to E(\mathcal{B})$$

is an equivalence.

In the next section we will consider functors $E: Cat_\infty^{ex} \to \mathcal{E}$ which satisfy condition ii) above. Such functors are said to be *additive.*

Example 4.2. The functor $K: Cat_\infty^{perf} \to Sp$ is a spectrum-valued additive invariant.

Note that the functor

$$(-)^{\simeq}: Cat_\infty^{ex} \to \mathcal{S}$$

which sends a small stable ∞-category $\mathcal{C}$ to the maximal subgroupoid $\mathcal{C}^{\simeq}$, is corepresented by the category Sp^{ω} of finite spectra. The same is true on Cat_∞^{perf}, since it is a full subcategory of Cat_∞^{ex}. The equivalence

$$Fun^{ex}(Sp^{\omega}, \mathcal{C})^{\simeq} \to \mathcal{C}^{\simeq}$$

is given by evaluation at the generator S^0. We will work with mapping spectra and write $\underline{Nat}(F, G)$ for the spectrum of natural transformations between functors $F, G: Cat_\infty^{perf} \to Sp$. The universal property of algebraic K-theory takes the following form:

Theorem 4.3 (Blumberg-Gepner-Tabuada)**.** *Let $E: Cat_\infty^{perf} \to Sp$ be an additive invariant. Then there is a natural equivalence of spectra*

$$\underline{Nat}(K, E) \simeq E(Sp^{\omega}).$$

Moreover, for any natural transformation $\alpha: \Sigma^\infty(-)^{\simeq} \to E$ there is an essentially unique natural transformation $\tilde{\alpha}: K \to E$ such that the diagram

$$\begin{array}{ccc} \Sigma^\infty(-)^{\simeq}_{+} & \xrightarrow{\alpha} & E \\ & {\scriptstyle\chi}\searrow \quad \nearrow{\scriptstyle\tilde{\alpha}} & \\ & K & \end{array}$$

commutes.

The two statements in the theorem above are in fact equivalent, as can be seen from the following commutative diagram:

$$\begin{array}{ccc} \underline{Nat}(\Sigma^\infty(-)^{\simeq}_{+}, E) & \xrightarrow{\simeq} & \underline{Nat}(\Sigma^\infty Map_{Cat_\infty^{ex}}(Sp^{\omega}, -)_{+}, E) \\ {\scriptstyle\chi^*}\uparrow & & \downarrow{\scriptstyle\simeq} \\ \underline{Nat}(K, E) & \xrightarrow{\simeq} & E(Sp^{\omega}) \end{array}$$

The top horizontal map is an equivalence by corepresentability and the right hand vertical map is an equivalence by the (spectrally enriched) Yoneda lemma. It follows that χ^* is an equivalence if and only if the lower horizontal map is, and that is the first map of the thoerem.

5 Relative K-theory and localization

We will now use simplicial techniques to build a model for the cofiber of the map induced on K-theory spectra by an exact functor $f: \mathcal{A} \to \mathcal{B}$.

Define the shift functor $Sh: \Delta \to \Delta$ on objects by $Sh([n]) = [1+n]$ and on morphisms by $Sh(\alpha)(0) = 0$ and $Sh(\alpha)(1+i) = 1 + \alpha(i)$. For a simplicial object $X_\bullet: \Delta^{op} \to \mathcal{E}$ we define

$$PX_\bullet := X_\bullet \circ (Sh)^{op}.$$

There is natural transformation $\delta^0: Id_\Delta \to Sh$ given by $\delta^0: [n] \to [1+n]$. This induces a natural map $d_0: PX_\bullet \to X_\bullet$ which we will apply to the S-construction. For us the most important fact about $PX_\bullet$ is the following:

Lemma 5.1. *Let $\mathcal{E}$ be an ∞-category and let $X_\bullet: \Delta^{op} \to \mathcal{E}$ be a diagram. Then there is a natural equivalence $|PX_\bullet| \xrightarrow{\simeq} X_0$.*

Definition 5.2. Let $f: \mathcal{A} \to \mathcal{B}$ be an exact functor between small stable ∞-categories. Define $Gap_\bullet(f: \mathcal{A} \to \mathcal{B})$ to be the pullback in the diagram

$$\begin{array}{ccc} Gap_\bullet(f: \mathcal{A} \to \mathcal{B}) & \longrightarrow & PGap_\bullet(\mathcal{B}) \\ \big\downarrow & & \big\downarrow \\ Gap_\bullet(\mathcal{A}) & \longrightarrow & Gap_\bullet(\mathcal{B}). \end{array}$$

Proposition 5.3. *An exact functor $f: \mathcal{A} \to \mathcal{B}$ between small stable ∞-categories induces a fiber sequence of spaces*

$$|S_\bullet(\mathcal{B})| \to \|S_\bullet Gap_\bullet(f: \mathcal{A} \to \mathcal{B}))\| \to \|S_\bullet Gap_\bullet \mathcal{A}\|.$$

The proof of this proposition which is given below extends easily to additive functors E with values in a presentable stable ∞-category $\mathcal{E}$. We write $|X_\bullet|$ to mean the (homotopy) colimit of a diagram $X_\bullet: \Delta^{op} \to \mathcal{E}$.

Proposition 5.4. *Let $\mathcal{E}$ be a presentable stable ∞-category and let $E: Cat^{ex} \to \mathcal{E}$ be an additive functor. Then any exact functor $f: \mathcal{A} \to \mathcal{B}$ between small stable ∞-categories induces a cofiber sequence in $\mathcal{E}$*

$$E(\mathcal{B}) \to |E(Gap_\bullet(f: \mathcal{A} \to \mathcal{B}))| \to |E(Gap_\bullet \mathcal{A})|$$

Proof of 5.3. For each $n \geq 0$ the vertical functors in the diagram

$$\begin{array}{ccc} Gap_n(f: \mathcal{A} \to \mathcal{B}) & \longrightarrow & Gap_n(\mathcal{B}) \\ \big\downarrow & & \big\downarrow \\ Gap_{1+n}(\mathcal{A}) & \longrightarrow & Gap_n(\mathcal{B}) \end{array}$$

are split-exact projections with kernels equivalent to $\mathcal{B}$. Applying $|S_\bullet(-)|$ to the diagram above gives a cartesian diagram for each $n \geq 0$, so we have a diagram of simplicial spaces

$$\begin{array}{ccc} ([n] \mapsto |S_\bullet Gap_n(f: \mathcal{A} \to \mathcal{B})|) & \longrightarrow & ([n] \mapsto |S_\bullet Gap_{1+n}(\mathcal{B})|) \\ \big\downarrow & & \big\downarrow \\ ([n] \mapsto |S_\bullet Gap_n(\mathcal{A})|) & \longrightarrow & ([n] \mapsto |S_\bullet Gap_n(\mathcal{B})|) \end{array}$$

which is levelwise cartesian and whose lower left hand space is connected for each $n \geq 0$. By a theorem of Bousfield-Friedlander, which was later improved by Rezk in [Rez01, Proposition 5.4] the induced diagram of geometric realizations

$$\begin{array}{ccc} \|S_\bullet Gap_\bullet(f: \mathcal{A} \to \mathcal{B})\| & \longrightarrow & \|S_\bullet PGap_\bullet(\mathcal{B})\| \\ \downarrow & & \downarrow \\ \|S_\bullet Gap_\bullet(\mathcal{A})\| & \longrightarrow & \|S_\bullet Gap_\bullet(\mathcal{B})\| \end{array}$$

is cartesian as well. By Lemma 5.1 the space $\|S_\bullet PGap_\bullet(\mathcal{B})\|$ is contractible and the vertical fibers are equivalent to $|S_\bullet \mathcal{B}|$. ☐

Applying Proposition 5.3 to the functor $Id_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ gives the following results:

Corollary 5.5. *For any small stable ∞-category $\mathcal{A}$ there is a natural equivalence of spectra*

$$K(\mathcal{A}) \xrightarrow{\simeq} \Omega|K(Gap_\bullet(\mathcal{A}))|.$$

Corollary 5.6. *Let $\mathcal{E}$ be a presentable stable ∞-category and let $E: Cat^{ex} \to \mathcal{E}$ be an additive functor. Then for any small stable ∞-category $\mathcal{A}$ there is a natural equivalence*

$$E(\mathcal{A}) \xrightarrow{\simeq} \Omega|E(Gap_\bullet(\mathcal{A}))|.$$

By extending the fiber sequence of Proposition 5.3 we see that we have found a model for relative algebraic K-theory:

Corollary 5.7. *An exact functor $f: \mathcal{A} \to \mathcal{B}$ between small stable ∞-categories induces a cofiber sequence of spectra*

$$K(\mathcal{A}) \to K(\mathcal{B}) \to |K(Gap_\bullet(f: \mathcal{A} \to \mathcal{B})|.$$

Theorem 5.8. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ be a strict-exact sequence of small stable ∞-categories. Then the induced sequence of K-theory spectra*

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$$

is a cofiber sequence.

To prove the theorem we need the notion of the nerve of an ∞-category. Considering a poset as an ∞-category gives a fully faithful functor

$$\Delta^\bullet: \Delta \to Cat_\infty.$$

Restricting the mapping space functor $Fun(-,-)^\simeq$ along this map induces a functor

$$\underline{N}_\bullet: Cat_\infty \to \mathcal{S}^{\Delta^{op}}$$

given by $\underline{N}_n(\mathcal{C}) \coloneqq Fun(\Delta^n, \mathcal{C})^\simeq$. This functor is fully faithful and its essential image is the full subcategory $CSS \subseteq \mathcal{S}^{\Delta^{op}}$ of complete Segal spaces.

Proof. Let $w\mathcal{B}$ denote the subcategory of $\mathcal{B}$ defined as the pullback in the diagram

$$\begin{array}{ccc} w\mathcal{B} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow g \\ \mathcal{C}^{\simeq} & \longrightarrow & \mathcal{C}, \end{array}$$

so that a morphism in $\mathcal{B}$ is in $w\mathcal{B}$ precisely if its cofiber is in $\mathcal{A}$.

For each $m \geq 0$ we write $Fun_w(\Delta^m, \mathcal{B})$ for the full subcategory of $Fun(\Delta^n, (\mathcal{B}))$ with objects those diagrams $\Delta^m \to \mathcal{B}$ such that the restriction to any $\Delta^1 \subset \Delta^m$ is a map in $w\mathcal{B}$. We also write $Fun_{\simeq}(\Delta^m, \mathcal{C})$ for the full subcategory of $Fun(\Delta^n, (\mathcal{C}))$ with objects those diagrams $\Delta^m \to \mathcal{C}$ such that the restriction to any $\Delta^1 \subset \Delta^m$ is a map in $\mathcal{C}^{\simeq}$. We define $wGap_n(\mathcal{B})$ and $Fun_w(\Delta^m, Gap_n(\mathcal{B}))$ as well as $Fun_{\simeq}(\Delta^m, Gap_n(\mathcal{C}))$ in an analogous way.

Forgetting filtration quotients gives an equivalence of categories

$$Gap_{1+n}(\mathcal{B})) \to Fun(\Delta^n, \mathcal{B}),$$

which restricts to an equivalence of categories

$$Gap_n(f{:}\mathcal{A} \to \mathcal{B}) \xrightarrow{\simeq} Fun_w(\Delta^n, \mathcal{B}).$$

Since both source and target are stable we can apply the Gap functor again to get a level equivalence of bisimplicial objects in Cat_∞^{ex} :

$$Gap_\bullet Gap_\bullet(f{:}\mathcal{A} \to \mathcal{B}) \to Gap_\bullet Fun_w(\Delta^\bullet, \mathcal{B}).$$

We will now analyse the right hand object. For each n there is an equivalence of simplicial stable ∞-categories

$$Gap_n Fun_w(\Delta^\bullet, \mathcal{B}) \simeq Fun_w(\Delta^\bullet, Gap_n(\mathcal{B})).$$

The functor $g{:}\mathcal{B} \to \mathcal{C}$ induces a map

$$Fun_w(\Delta^\bullet, Gap_n(\mathcal{B})) \to Fun_{\simeq}(\Delta^\bullet, \mathcal{C}),$$

of simplicial stable ∞-categories. After taking maximal subgroupoids this can be identified with the map of complete segal spaces

$$\underline{N}_\bullet(wGap_n(\mathcal{B})) \to \underline{N}_\bullet(Gap_n(\mathcal{C})^{\simeq})$$

which corresponds to the map $wGap_n(\mathcal{B}) \to Gap_n(\mathcal{C}))^{\simeq}$ in Cat_∞, which is a localization and hence cofinal by [Cis19, Cor. 7.6.9].

I claim that the maps in the induced sequence bisimplicial spaces become equivalences up taking geometric realizations:

$$S_\bullet Gap_\bullet(f{:}\mathcal{A} \to \mathcal{B}) \to Fun_w(\Delta^\bullet, Gap_\bullet(\mathcal{B}))^{\simeq} \to \underline{N}_\bullet(Gap_\bullet(\mathcal{C})^{\simeq}).$$

The left hand map is a pointwise equivalence by the arguments above and therefore induces an equivalence on geometric realizations. For each $n \geq 0$ the geometric realization of the map $\underline{N}_\bullet(wGap_n(\mathcal{B})) \to \underline{N}_\bullet(Gap_n(\mathcal{C})^{\simeq})$ is given by the corresponding map $|wGap_n(\mathcal{B})| \to |Gap_n(\mathcal{C}))^{\simeq}|$

which is an equivalence. This is because the geometric realization of complete Segal spaces is given by the geometric realization of the corresponding quasi-categories, by [Bar13, 3.9], and in this case that map is cofinal. Since Δ^{op} is sifted we may take geometric realization (=colimit) first in the nerve direction, which gives a pointwise equivalence of simplicial spaces, and then in the $Gap_\bullet$-direction to get an equivalence of spaces.

Using [Bar13, 3.9] again we see that the natural map $|S_\bullet \mathcal{C}| \to |\underline{N}_\bullet(Gap_\bullet(\mathcal{C})^{\simeq})|$ is an equivalence, so all in all we have a natural equivalence

$$|S_\bullet Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})| \simeq |S_\bullet \mathcal{C}|.$$

Looping this and applying Corollary 5.7 now gives the result.

□

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