Universality and localization for K-theory

Kristian J. Moi

KTH, Stockholm

KDAG seminar
April 15, 2020

Two results today

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Universal property

K-theory is the initial “additive invariant” with a natural map $(\mathcal{C})^{\simeq} \to K(\mathcal{C})$ of spaces

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Localization theorem

K-theory takes “strict-exact” sequences

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

of small stable ∞-categories to cofiber sequences of spectra.

Notation

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Exact sequences of stable ∞-categories

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$E\colon Cat_\infty^{perf} \to \mathcal{E}$ which is additive and commutes with filtered colimits is called an additive invariant.

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Example

The functor $K\colon Cat_\infty^{perf} \to Sp$ is a spectrum-valued additive invariant.

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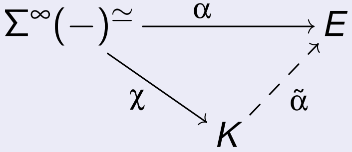

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$$\begin{array}{ccc} \Sigma^\infty(-)^\simeq & \xrightarrow{\alpha} & E \\ & {\scriptstyle \chi}\searrow & \nearrow{\scriptstyle \tilde{\alpha}} \\ & K & \end{array}$$

has a unique filler $\tilde{\alpha}$.

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$$\begin{array}{ccc} Nat(\Sigma^\infty(-)^{\simeq}_{+}, E) & \xrightarrow{\simeq} & Nat(\Sigma^\infty Map_{Cat_\infty^{ex}}(Sp^{fin}, -)_+, E) \\ \chi^* \uparrow & & \downarrow \simeq \\ Nat(K, E) & \xrightarrow{\simeq} & E(Sp^{fin}) \end{array}$$

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Conclude: χ^* is an equivalence.

Simplicial tools

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Lemma

Let $\mathcal{E}$ be an ∞-category and $X_\bullet\colon \Delta^{op} \to \mathcal{E}$ a diagram.

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Lemma

Let $\mathcal{E}$ be an ∞-category and $X_\bullet\colon \Delta^{op} \to \mathcal{E}$ a diagram.Then there is a natural equivalence $|PX_\bullet| \xrightarrow{\simeq} X_0$.

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Proposition

For any $f\colon \mathcal{A} \to \mathcal{B}$ exact functor, there is an induced cofibre sequence of spectra

$$K(\mathcal{A}) \to K(\mathcal{B}) \to |K(Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})|.$$

Proof for $Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})$

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1. For each $n \geq 0$ have diagram

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Proof for $Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})$

1. Apply $K(-)$ levelwise to get

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Additivity $\implies$ Cartesian $\implies$ cocartesian

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4. Loop to get cofiber sequence

$$K(\mathcal{A}) \to K(\mathcal{B}) \to |K(Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})|.$$

The localization theorem

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Theorem

A strict-exact sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ *in* Cat_∞^{ex}

The localization theorem

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A strict-exact sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathrm{Cat}_{\infty}^{ex}$ induces a cofiber sequence of spectra

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Example

The sequence of spectra $K(Perf_{tors}(\mathbb{Z})) \to K(\mathbb{Z}) \to K(\mathbb{Q})$ is a cofiber sequence.

Proof of localization

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- Conclude: $|K(Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})| \xrightarrow{\simeq} \Omega|Fun_\simeq(\Delta^\bullet, Gap_\bullet(\mathcal{C}))|$

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We show $|K(Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})| \simeq K(\mathcal{C})$.

- $Gap_n(f\colon \mathcal{A} \to \mathcal{B}) \xrightarrow{\simeq} Fun_w(\Delta^n, \mathcal{B})$ where Fun_w means maps in Δ^n go to maps with cofiber in $\mathcal{A}$
- Level equivalence $S_n Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B}) \to Fun_w(\Delta^\bullet, Gap_n(\mathcal{B}))^\simeq$
- $Fun_w(\Delta^\bullet, \mathcal{B})^\simeq \to Fun_\simeq(\Delta^\bullet, \mathcal{C})^\simeq$ equivalence on $|-|$ (model for $w\mathcal{B} \to i\mathcal{C}$ which is localization)
- Conclude: $|K(Gap_\bullet(f\colon \mathcal{A} \to \mathcal{B})| \xrightarrow{\simeq} \Omega|Fun_\simeq(\Delta^\bullet, Gap_\bullet(\mathcal{C}))|$ and $\Omega|Fun_\simeq(\Delta^\bullet, Gap_\bullet(\mathcal{C}))| \xleftarrow{\simeq} K(\mathcal{C})$

Thank you for you attention!