Connections to classical K-theory

Erik Lindell

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1. K-theory of exact 1-categories

1.1 Exact 1-categories

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In our applications today we will only consider abelian categories.

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- In algebraic geometry, we define $K(X) := K(\mathrm{Vect}(X))$ and $G(X) := K(\mathrm{QCoh}(X))$, for a scheme X.

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such that $0 \to A_{i,j} \to A_{i,k} \to A_{j,k} \to 0$ is exact, for all $0 \leq i \leq j \leq k \leq n$. The morphisms are isomorphisms of diagrams. The face (degeneracy) maps are given by deleting (inserting) rows and columns. The K-theory space of $\mathcal{C}$ is now defined by $K(\mathcal{C}) := \Omega|BS_n\mathcal{C}|$.

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- For example, the subcategory of cofibrant objects in a model category is a Waldhausen category.
- An exact category is a Waldhausen category where the weak equivalences are the isomorphisms and the cofibration sequences are the exact sequences.

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Thus $[B_i] = [B_i/B_{i+1}] + [B_{i+1}]$ in $K_0(\mathcal{B})$, so

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and since each quotient lies in $\mathcal{A}$, the homomorphism is surjective. $\square$

2. Theorem of the Heart

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We use the notation $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$.

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- If $\mathcal{C}$ is equipped with a *t*-structure, we denote by $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ the full subcategories of $\mathcal{C}$ spanned by $(\mathrm{h}\mathcal{C})_{\geq n}$ and $(\mathrm{h}\mathcal{C})_{\leq n}$, respectively.

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- We say that a t-structure on $\mathcal{C}$ is *bounded* if

$$\mathcal{C} = \left(\bigcup_{n \geq 0} \mathcal{C}_{\leq n} \right) \cap \left(\bigcup_{n \geq 0} \mathcal{C}_{\leq -n} \right)$$

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If $\mathcal{A}$ is a stable ∞-category that admits a bounded t-structure, then the inclusion $\mathrm{C}^{\heartsuit} \subset \mathcal{C}$ induces a weak equivalence

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3. Applications

Recall, from Kristian's talk, that we have a strict-exact sequence

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3.1. The localization $\mathbb{Z} \to \mathbb{Q}$

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of spectra. We will now apply our tools to compute the homotopy groups of $K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z}))$.

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- We have a t-structure on $\mathcal{C} = \mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$, where $\mathcal{C}_{\geq 0}$ (respectively $\mathcal{C}_{\leq 0}$) is generated by complexes with homology concentrated in non-negative (resp. non-positive) degree.

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Thus

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3.1. The localization $\mathbb{Z} \to \mathbb{Q}$

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We have an equivalence $\mathsf{FinAb} \cong \prod_{p \text{ prime}} \mathsf{FinAb}_p$, so

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where each quotient is a product of $\mathbb{F}_p$'s. Thus

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \simeq \prod_{p \text{ prime}} K(\mathbb{F}_p).$$

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Corollary

For $n > 0$, the K-groups of $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$ are given by

$$K_n(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) = \begin{cases} \prod_{p \text{ prime}} \mathbb{Z}/(p^n - 1)\mathbb{Z} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Thank you for listening!