Classical results in K-theory

Erik Lindell

Contents

1 Introduction

These are (somewhat) expanded lecture notes based on a talk given in the K-theory and derived algebraic geometry seminar at KTH and Stockholm University.

The focus of these notes are two classical results in K-theory. The first is Quillen's *Dévissage theorem*, which is a result in [Quillen72] concerning the K-theory of abelian categories. Before we state the theorem we will define *exact* categories, which is the framework in which Quillen worked. We will then show how their K-theory can be constructed in analogy to what we saw in the talk on the $S_\bullet$-construction for stable ∞-categories. Finally, we will state Quillen's Dévissage theorem. We will not prove this theorem in general, but only in the much simpler case for K_0, to demonstrate how the assumptions of the theorem can be used.

The second result we are interested in is the much more recent *Theorem of the Heart*, which is due to Barwick in [Barwick15] (originally uploaded as a preprint in 2012). This is a result about stable ∞-categories which admit bounded t-structures. We will therefore start by recalling some basic definitions concerning t-structures on stable ∞-categories, in order to be able to state the theorem.

To conclude we will apply these theorems in conjunction, in order to compute the K-groups of the category $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$, of perfect complexes over $\mathbb{Z}$ with homology groups which are all torsion. We saw this category previously, in the talk on universality and localization for K-theory, were it appeared in the cofiber sequence

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \to K(\mathbb{Z}) \to K(\mathbb{Q})$$

of spectra.

2 K-theory of exact 1-categories

2.1 Exact 1-categories

Exact categories were originally introduced by Quillen in [Quillen72], to capture the concept of a category which admit a notion of exact sequences. The definition we use here is a bit different from Quillen's original definition, and is taken from [Weibel13]. Here an exact category is more or less defined as a subcategory of an abelian category which is closed under extensions. More precisely:

Definition 1. An exact category is a pair $(\mathcal{C}, \mathcal{E})$, of an additive category $\mathcal{C}$ and a class $\mathcal{E}$ of sequences in $\mathcal{C}$ of the form

$$0 \to A \to B \to C \to 0, \tag{$*$}$$

such that there exists an embedding $\mathcal{C} \hookrightarrow \mathcal{A}$ of $\mathcal{C}$ as a full subcategory of an abelian $\mathcal{A}$ category so that

1. $\mathcal{E}$ is the class of all sequences in $\mathcal{C}$ which are exact in $\mathcal{A}$,
2. if $(*)$ is an exact sequence in $\mathcal{A}$ with $A, C \in \mathcal{C}$, then B also lies in $\mathcal{C}$.

In our applications we will really only consider abelian categories, but since there are many important categories which are exact but not abelian, it is good to note that the construction of K-theory we demonstrate here work more generally than for abelian categories, even in the 1-categorical case.

Example 2. A primary example of interest in K-theory is $\mathrm{Mod}^{\mathrm{proj}}(R))$, the category of finitely generated and projective R-modules. This category is abelian if R is a ring of global dimension zero, but only exact otherwise. The K-theory of this category is commonly called the *K-theory of the ring R*.

Example 3. Another natural category to consider is $\mathrm{Mod}(R)$, the category of finitely generated R-modules. This category is always exact, but only abelian if R is Noetherian. The K-theory of this category is commonly called the *G-theory of the ring R*, and we write $G(R) := K(\mathrm{Mod}(R))$. If R is Noetherian and regular the G- and K-theory of R agree.

Example 4. In algebraic geometry, the analogy to Example 2 is the category $\mathrm{Vect}(X)$, of vector bundles on a scheme X, which is exact as a subcategory of the category of the category of all $\mathcal{O}_X$-modules. The K-theory of this category is usually called the K-theory *of the scheme* X and is denoted $K(X)$.

The analogy to Example 3 is the category $\mathrm{Coh}(X)$, of coherent sheaves on X. This category is always exact, but whether it is abelian depends on the definition of coherent sheaf that is used. With the definition used in [Weibel13], the existence of kernels is generally not guaranteed. If X is a Noetherian scheme, this problem is fixed and the category does indeed become abelian. Another common definition of coherent shaves, which is for example used in Grothendieck's EGA 1 [Grothendieck60], adds a condition which guarantees the existence of kernels and which makes the category abelian for any scheme X.

In the same vain as in Example 3, the K-theory of this category is called the G-theory of X and is denoted $G(X)$. For sufficiently nice schemes, such as smooth varieties, we have $K(X) \simeq G(X)$.

2.2 The $S_\bullet$-construction for exact 1-categories

The K-theory of exact 1-categories can be defined using Waldhausen's $S_\bullet$-construction, completely analogously to what we have seen in a previous talk.

Definition 5. We construct a simplicial category $S_\bullet\mathcal{C}$, where the objects of $S_n\mathcal{C}$ are diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{0,1} & \longrightarrow & A_{0,2} & \longrightarrow & \cdots & \longrightarrow & A_{0,n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & \longrightarrow & A_{1,2} & \longrightarrow & \cdots & \longrightarrow & A_{1,n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & 0 & \longrightarrow & \cdots & \longrightarrow & A_{2,n} \\
 & & & & & & & & \vdots \\
 & & & & & & & & 0
\end{array}$$

such that $0 \to A_{i,j} \to A_{i,k} \to A_{j,k} \to 0$ is exact, for all $0 \leqslant i \leqslant j \leqslant k \leqslant n$. The morphisms of $S_n\mathcal{C}$ are isomorphisms of diagrams. The face map ∂_i is defined by deleting the ith row and column, and the degeneracy maps are instead defined by inserting rows and columns in the corresponding way. The K-theory space of $\mathcal{C}$ is now defined as

$$K(\mathcal{C}) := \Omega|BS_n\mathcal{C}|.$$

Remark 6. In the analogous construction for stable ∞-category, we defined $S_n\mathcal{C}$ by first constructing an ∞-category $\mathrm{Gap}_{[n]}(\mathcal{C})$, whose objects were the diagrams as defined above. There we required that for all $0 \leqslant i \leqslant j \leqslant k \leqslant n$, the diagram

$$\begin{array}{ccc}
A_{i,j} & \longrightarrow & A_{i,k} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A_{i,k}
\end{array}$$

was a push-out square. This is analagous to the exactness assumption in this construction. There we also defined $S_n\mathcal{C}$ by restricting $\mathrm{Gap}_{[n]}(\mathcal{C})$ to a maximal Kan complex, which is analogous to restricting morphisms to isomorphisms of diagrams in this construction.

Remark 7. When Quillen introduced the K-theory of exact 1-categories, he did not use this construction. His construction, called the Q-construction, can for example be found in [Quillen72] and [Weibel13]. The $S_\bullet$-construction was introduced by Waldhausen in order to define K-theory of a more general class of 1-categories, which are now called *Waldhausen categories*. These are a certain kind of 1-categories with a notion of cofibrations and weak equivalences. An exact category is a Waldhausen category where the weak equivalences are the isomorphisms and the cofibration sequences are the exact sequences. To give a more general example, the subcategory of cofibrant objects in a model category is always Waldhausen category. For example, we thus see that the category of CW-complexes with the Quillen model structure is a Waldhausen category.

2.3 Dévissage

We end the first part with our first theorem of interest:

Theorem 8. (*Dévissage*) Let $\mathcal{A} \subset \mathcal{B}$ be an exact abelian subcategory of an abelian category, which is closed in $\mathcal{B}$ under subobjects and quotients. If every object $B \in \mathcal{B}$ admits a finite filtration

$$0 = B_r \subset B_{r-1} \subset \cdots \subset B_0 = B,$$

such that each quotient B_i/B_{i+1} lies in $\mathcal{A}$, then

$$K(\mathcal{A}) \simeq K(\mathcal{B}).$$

Proof for K_0. We have a homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by mapping $[A] \mapsto [A]$, for the class of any object A in $\mathcal{A}$. We start by proving that this homomorphism is injective. Suppose A is an object of $\mathcal{A}$ such that $[A] = 0$ in $K_0(\mathcal{B})$. This means that the unique map $A \to 0$ in $\mathcal{B}$ is an isomorphism, where we use 0 to denote a zero object in $\mathcal{B}$. By composing with this isomorphism (and its inverse, respectively), we see that every object $A' \in \mathcal{A}$ has a unique morphism $A \to A'$ and a unique morphism $A' \to A$. Since $\mathcal{A} \subset \mathcal{B}$ is a full subcategory by virtue of being exact, it follows that A is already a zero object in $\mathcal{A}$ and so $[A] = 0$ in $K_0(\mathcal{A})$.

To prove surjectivity, we let B be an object of $\mathcal{B}$ and

$$0 = B_r \subset B_{r-1} \subset \cdots \subset B_0 = B,$$

a filtration as in the theorem. For each i we get a short exact sequence

$$0 \to B_{i+1} \to B_i \to B_i/B_{i+1} \to 0.$$

Thus $[B_i] = [B_i/B_{i+1}] + [B_{i+1}]$ in $K_0(\mathcal{B})$, so

$$[B] = \sum_{i=0}^{r} [B_i/B_{i+1}],$$

and since each quotient lies in $\mathcal{A}$, the homomorphism is surjective. $\square$

3 The Theorem of the Heart

To be able to state the theorem, we will need some basic definitions from the theory of t-structures on stable ∞-categories, so let us shortly recall some of the things we learned in the lecture on stable ∞-categories.

3.1 Refresher on t-structures

A t-structure is classically defined on a triangulated category.

Definition 9. A t-structure on a triangulated category $\mathcal{D}$ is a pair $\mathcal{D}_{\geqslant 0}$, $\mathcal{D}_{\leqslant 0}$ of full subcategories such that

1. $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$ for all $X \in \mathcal{D}_{\geqslant 0}$ and $Y \in \mathcal{D}_{\leqslant 0}$,
2. $\mathcal{D}_{\geqslant 0}[1] \subseteq \mathcal{D}_{\geqslant 0}$ and $\mathcal{D}_{\leqslant 0}[-1] \subseteq \mathcal{D}_{\leqslant 0}$
3. and for any $X \in \mathcal{D}$, there exists a distinguished triangle $X' \to X \to X''$ with $X' \in \mathcal{D}_{\geqslant 0}$ and $X'' \in \mathcal{D}_{\leqslant 0}[-1]$.

We will use the notation $\mathcal{D}_{\geqslant n} := \mathcal{D}_{\geqslant 0}[n]$ and $\mathcal{D}_{\leqslant n} := \mathcal{D}_{\leqslant 0}[n]$.

We now move back to the ∞-categorical world, so let $\mathcal{C}$ be stable ∞-category. Recall that the homotopy category $\mathrm{h}\mathcal{C}$ admits a natural triangulated structure, where the distinguished triangles come from pushout squares of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Definition 10. We define a t-structure on $\mathcal{C}$ to be a t-structure on its homotopy category.

- If $\mathcal{C}$ is equipped with a t-structure, we denote by $\mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n}$ the full subcategories of $\mathcal{C}$ spanned by $(\mathrm{h}\mathcal{C})_{\geqslant n}$ and $(\mathrm{h}\mathcal{C})_{\leqslant n}$, respectively.
- If $\mathcal{C}$ is equipped with a t-structure, we define its *heart* to be the full subcategory $C^{\heartsuit} := \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$. The heart is equivalent to the nerve of its homotopy category, which is always abelian.
- We say that a t-structure on $\mathcal{C}$ is *bounded* if

$$\mathcal{C} = \left(\bigcup_{n \geqslant 0} \mathcal{C}_{\leqslant n} \right) \cap \left(\bigcup_{n \geqslant 0} \mathcal{C}_{\geqslant -n} \right)$$

3.2 Statement of the theorem

We now have all we need to state our second theorem of interest.

Theorem 11. *(Theorem of the Heart)* If $\mathcal{A}$ is a stable ∞-category that admits a bounded t-structure, then the inclusion $\mathsf{C}^{\heartsuit} \subset \mathcal{C}$ induces a weak equivalence

$$K(\mathsf{C}^{\heartsuit}) \xrightarrow{\simeq} K(\mathcal{C}).$$

Remark 1. This is one of few known results in K-theory that can be used to derive equivalences on K-theory that do not arise from equivalences of the ∞-categories themselves. The other most imporant example is *Dévissage.*

Remark 2. Barwick's Theorem of the Heart has an important predecessor in *Neeman's* Theorem of the Heart, originally from [Neeman98], which is an analogous theorem for the K-theory of triangulated categories. However, this theorem gives no a priori reason to believe that Barwick's Theorem of the Heart should hold. In fact, there are examples of model categories with equivalent homotopy categories, but different Waldhausen K-theories. The original such example was due to Schlichting in [Schlichting02]. Therefore, Neeman's K-theory of triangulated categories cannot be used to draw conclusions about Waldhausen K-theory.

4 An application

We are now in shape to use our results to compute some K-theory. In the talk on universality and localization of K-theory, we saw how the localization map $\mathbb{Z} \to \mathbb{Q}$ gives rise to a so-called *strict exact* sequence $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Q})$, i.e. a pushout square

$$\begin{array}{ccc} \mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z}) & \longrightarrow & \mathrm{Perf}(\mathbb{Z}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Perf}(\mathbb{Q}) \end{array}$$

which is both cartesian and cocartesian in $\mathrm{Cat}_\infty^{\mathrm{ex}}$, the ∞-category of small stable ∞-categories and exact functors. We then saw how the localization theorem, applied to this sequence, gives rise to a cofiber sequence

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \to K(\mathbb{Z}) \to K(\mathbb{Q})$$

of K-theory spectra, where $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$ is the ∞-category of chain complexes over $\mathbb{Z}$ whose homology groups all only consist of torsion. We will now apply our tools to compute the homotopy groups of this category. This will be done in three steps.

The first step is to apply the Theorem of the Heart. The category $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$ has a t-structure, where $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})_{\geqslant 0}$ is generated by the chain complexes with homology concentrated in non-negative degree. Similarly, $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})_{\leqslant 0}$ is generated by the chain complexes with homology concentrated in non-positive degree. Recall that a perfect chain complex is a chain complex whose homology is bounded. Hence this t-structure on $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$ is bounded, so the Theorem of the Heart is applicable. Furthermore, the heart consists of those chain complexes whose homology is concentrated in degree zero. Since the homology is required to only consist of torsion, it follows that we have an equivalence $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})^{\heartsuit} \cong \mathrm{FinAb}$, where FinAb denotes the category of finite abelian groups. By applying the Theorem of the Heart we thus get

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \simeq K(\mathsf{FinAb}).$$

The second step is to factorize. We have an equivalence $\mathsf{FinAb} \cong \bigoplus_{p \text{ prime}} \mathsf{FinAb}_p$, where $\bigoplus p\ \mathrm{prime}\mathsf{FinAb}_p$ is the *restricted product* (or *direct sum*) of the categories of finite p-groups, for all primes p. The restricted sum is defined as the filtered colimit over all finite partial products. Since K-theory commutes with finite products and with filtered colimits, it follows that

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \cong \bigoplus_{p \text{ prime}} K(\mathsf{FinAb}_p),$$

where we now use $\bigoplus$ to denote the restricted product of spaces.

The third and final step is to apply Dévissage. The category $\mathrm{Mod}^{\mathrm{proj}}(\mathbb{F}_p) \cong \mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fin}}$, of finite dimensional $\mathbb{F}_p$-vector spaces, is an exact abelian subcategory of FinAb_p, which is closed under subobjects and quotients. A finite abelian p-group A has the form

$$A \cong \prod_{k=1}^{n} \mathbb{Z}/p^{i_k}\mathbb{Z},$$

so each such group has a filtration

$$0 \leqslant \cdots p^r A \leqslant p^{r-1} \leqslant \cdots \leqslant pA \leqslant A,$$

for some large enough r. Here each quotient is a product of $\mathbb{F}_p$'s, so we can apply Dévissage to draw the conclusion that

$$K(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) \cong \bigoplus_{p \text{ prime}} K(\mathbb{F}_p).$$

Since $\mathbb{F}_p$ is a field, $K_0(\mathbb{F}_p) \cong \mathbb{Z}$. The higher K-groups of $\mathbb{F}_p$ is a classical result by Quillen, once again from the paper ['72]:

Theorem 12. For $n > 0$, we have

$$K_{2n-1}(\mathbb{F}_p) = \mathbb{Z}/(p^n - 1)\mathbb{Z}, \quad K_{2n}(\mathbb{F}_p) = 0.$$

Corollary 13. For $n > 0$, the K-groups of $\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})$ are given by

$$K_n(\mathrm{Perf}_{\mathrm{tors}}(\mathbb{Z})) = \begin{cases} \bigoplus_{p \text{ prime}} \mathbb{Z}/(p^n-1)\mathbb{Z} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

References

[Barwick15] Barwick, C. (2015). On exact ∞-categories and the Theorem of the Heart. Compositio Mathematica, 151(11), 2160-2186. doi:10.1112/S0010437X15007447

[Grothendieck60] Grothendieck, Alexander. Éléments de géométrie algébrique : I. Le langage des schémas. Publications Mathématiques de l'IHÉS, Volume 4 (1960) , pp. 5-228.

[Neeman98] Neeman, A. K-theory for triangulated categories. III(A). The theorem of the heart, Asian J. Math. 2 (1998), no. 3, 495-589. MR 1724625

[Quillen72] Quillen D. (1973) Higher algebraic K-theory: I. In: Bass H. (eds) Higher K-Theories. Lecture Notes in Mathematics, vol 341. Springer, Berlin, Heidelberg

[Schlichting02] M. Schlichting, A note on K-theory and triangulated categories, Invent. Math. 150 (2002), no. 1, 111-116. MR 1930883 (2003h:18015)

[Weibel13] Weibel C. (2013) The K-book. Graduate Studies in Math. vol. 145, AMS, 2013