Non-connective K-theory

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Todays plan

- An overview on exactness and additivity.
- Cofinality and Non-connective K-theory.
- K-theory of schemes.
- Localizing invariants and universal property.
- THH as a localizing invariant.

Exactness in $\mathrm{Cat}_\infty^{\mathrm{perf}} \subset \mathrm{Cat}_\infty^{\mathrm{ex}}$: $\mathrm{Idem}(-)$

A sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is called:

- **exact** if it is fibration and a cofibration.
- **strict-exact** if it is exact and the essential image of g is $\mathcal{C}$.
- **split-exact** if it is exact and g has a right adjoint r with co-unit $\varepsilon : g \circ r \xrightarrow{\simeq} \mathrm{Id}_{\mathcal{C}}$.

A sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is called:

- **exact** if $\mathrm{Idem}(-)$ makes it exact in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.
- **strict-exact** if it is a fibration and a cofibration.
- **split-exact** if it is exact and g has a right adjoint r with co-unit $\varepsilon : g \circ r \xrightarrow{\simeq} \mathrm{Id}_{\mathcal{C}}$.

Note: split-exact $\Rightarrow$ strict-exact $\Rightarrow$ exact. In all of these f is fully faithful and the idempotent completion of the essential image of g is $\mathcal{C}$.

Exactness in $\mathrm{Cat}_\infty^{\mathrm{perf}} \subset \mathrm{Cat}_\infty^{\mathrm{ex}}$: $\mathrm{Idem}(-)$

Reformulated:

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

In $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is exact if f is fully faithful and $\mathrm{Idem}(\mathcal{B}/\mathcal{A}) \simeq \mathcal{C}$, and strict exact if $\mathcal{B}/\mathcal{A} \simeq \mathcal{C}$.

In $\mathrm{Cat}_\infty^{\mathrm{ex}}$ strict-exactness means that

- f is fully faithful.
- The essential image of f is closed under retracts in $\mathcal{B}$.
- $\mathcal{B}/\mathcal{A} \simeq \mathcal{C}$ (implies that g is essentially surjective).

Additive invariant and universal property

A functor $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{E}$ (presentable stable) is called an **additive invariant** iff 1) it commutes with filtered colimits and 2) preserves split-exactness.

Example: $K(\mathcal{C}) = \Omega|S_\bullet \mathcal{C}|$. In fact (Kristian's talk 5.8),

$$K : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p$$

sends strict-exact to cofibrations. Beware though: This is not additive on $\mathrm{Cat}_\infty^{\mathrm{ex}}$ as defined in [BGT]. Indeed, to get something additive one should use $K(\mathrm{Idem}(-))$ (their K).

Theorem (Blumberg-Gepner-Tabuada)

For any additive invariant $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}p$ there is a natural equivalence

$$\begin{array}{ccc} \Sigma^\infty(-)^{\simeq} & \xrightarrow{\alpha} & E \\ & \searrow^{\chi} & \uparrow \overline{\alpha}\ \exists! \\ & & K \end{array}$$

$$\mathrm{Nat}(K, E) \simeq E(\mathcal{S}p^\omega).$$

Why Non-connective K?

Consider an exact $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$$

is not always a (co)fibration of spectra. Indeed, it may not be strict-exact meaning that g may not be essentially surjective (only after idempotent completion).

Lemma

This is never worse than $K_0(\mathcal{B}) \to K_0(\mathcal{C})$ not being surjective.

“Proof” on next slide.

Confinality theorem

A functor $f : \mathcal{A} \to \mathcal{B}$ is (classically) called **cofinal** if for each $b \in \mathcal{B}$ there is a $b' \in \mathcal{B}$ such that $b \oplus b' \simeq f(a)$ for some a.

Theorem

Let $f : \mathcal{A} \to \mathcal{B}$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ be fully faithful and cofinal. The induced map $f_* : K_n(\mathcal{A}) \to K_n(\mathcal{B})$ is an isomorphism for $n \geq 1$ and injective for $n = 0$.

- If the essential image of f is closed under retracts in $\mathcal{B}$ it also becomes essentially surjective, and hence an $\simeq$.
- This implies that for the inclusion $\mathcal{A} \to \mathrm{Idem}(\mathcal{A})$ we get that only K_0 differs.
- For an $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ exact we have that

$$\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$$

  is strict-exact (in $\mathrm{Cat}_\infty^{\mathrm{ex}}$) and hence the lemma on the previous slide follows.

Why exact in $\mathrm{Cat}_\infty^{\mathrm{perf}}$?

Recall the strict-exact sequence

$$\mathrm{Perf}_{tors}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{Q}).$$

However, the exact sequence

$$\mathrm{Perf}_S(R) \to \mathrm{Perf}(R) \to \mathrm{Perf}(S^{-1}R)$$

is not always strict exact. Example!?: Relative topological K-theory for (X, A):

$$\mathrm{Perf}_A(C(X)) \to \mathrm{Perf}(C(X)) \to \mathrm{Perf}(C(A))$$

Here $A \subset X$ is usually closed, but we can replace it by an equivalent neighborhood. Indeed,

$$\cdots \to K^{-1}(A) \to K^0(X/A) \to K^0(X) \to K^0(A)$$

is exact, but the last map is often not surjective.

Sketch of proof of cofinality

Defining non-connective K-theory

We now extend the already defined $K : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p_{\geq 0}$ to a functor $\mathbb{K} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p$ generally with image a non-connective spectrum.

Idea: Construct an $F\mathcal{A} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$ with $K(F\mathcal{A}) = 0$ and fully faithful $\mathcal{A} \subset F\mathcal{A}$ and define the K-theoretic suspension by

$$\Sigma\mathcal{A} = F\mathcal{A}/\mathcal{A}.$$

When $\mathcal{A} \to F\mathcal{A} \to \Sigma\mathcal{A}$ is strict-exact we get long exact sequence proving $K_{n+1}(\Sigma\mathcal{A}) \cong K_n(\mathcal{A})$. However, (as constructed by BGT) it is not strict-exact. Indeed, $F\mathcal{A} = \mathrm{Ind}_\kappa(\mathcal{A})$ (which has $K = 0$ by the Eilenberg Swindle) is idempotent complete. So, the inclusion can only be part of a strict exact sequence if $\mathcal{A} \simeq \mathrm{Idem}(\mathcal{A})$. But still, cofinality shows

$$K_1(\Sigma\mathcal{A}) \cong K_0(\mathrm{Idem}\,\mathcal{A}) \qquad \text{and} \qquad K_{n+1}(\Sigma\mathcal{A}) \cong K_n(\mathcal{A}), n \geq 1.$$

and we define $\mathbb{K}(\mathcal{A}) = \lim_{k\to\infty} \Omega^k K(\Sigma^k\mathcal{A})$.

Thomason-Trobaugh

Let X be a scheme and Z a closed sub-scheme. Define

$$\mathrm{Perf}(X \text{ on } Z)$$

as the infinity category of perfect complexes on X that are acyclic on U (hence supported "near" Z - with nice conditions "on" Z).

Theorem (Thomason-Trobaugh)

For X quasi-compact quasi-separable and $U \subset X$ Zariski open and quasi-compact (with $Z = X - U$) we have

$$\mathbb{K}(X \text{ on } Z) \to \mathbb{K}(X) \to \mathbb{K}(U)$$

is a (co)fibration in spectra.

Proof?

Is $\mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ exact? □

Excision

Theorem (Excision)

Let X, Z and U be as in the previous theorem and let $V \subset X$ be open with $Z \subset V$. We have an equivalence

$$\mathbb{K}(X \text{ } on \text{ } Z) \simeq \mathbb{K}(V \text{ } on \text{ } Z)$$

induced by the restriction map.

Proof?

I believe that the ∞-functor $j^* : \mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(V \text{ on } Z)$ is an equivalence, with inverse $j_!$ (extension by 0). □

Mayer-Vietoris

Theorem

With $X = U \cup V$ q.c. and q.s. and U and V open in X and q.c. the following diagram is cartesian:

$$\begin{array}{ccc} \mathbb{K}(X) & \longrightarrow & \mathbb{K}(U) \\ \downarrow & & \downarrow \\ \mathbb{K}(V) & \longrightarrow & \mathbb{K}(U \cap V) \end{array}$$

Proof.

Easy from excision and Thomason Trobaugh. The map of fibers is an equivalence:

$$\begin{array}{ccccc} \mathbb{K}(X \text{ on } Z) & \longrightarrow & \mathbb{K}(X) & \longrightarrow & \mathbb{K}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{K}(V \text{ on } Z) & \longrightarrow & \mathbb{K}(V) & \longrightarrow & \mathbb{K}(U \cap V) \end{array}$$

□

Bass' Definition

Bass originally defined negative K groups by:

$$K_i(R) = \operatorname{coker}(K_{i+1}(R[t]) \oplus K_{i+1}(R[t^{-1}]) \to K_{i+1}(R[t^{\pm 1}])).$$

This forces MV for (with $X = \operatorname{Spec} R$)

$$\begin{array}{ccc} \mathbb{K}(X \times \mathbb{P}^1) & \longrightarrow & \mathbb{K}(X \times \mathbb{A}^1) \\ \downarrow & & \downarrow \\ \mathbb{K}(X \times \mathbb{A}^1) & \longrightarrow & \mathbb{K}(X \times (\mathbb{A}^1)^*) \end{array}$$

Indeed, $K(X \times \mathbb{P}^1) \cong K(X) \oplus K(X)$ and $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$, and on K the two maps out of the top left corner are the same projection to one of the factors.

Weibel's Conjecture

Theorem (Kerz-Strunk-Tamme)

Let X be a Noetherian scheme of Krull dimension $d < \infty$ then:

- $\mathbb{K}_n(X) = 0$ *for* $n < -d$
- $\mathbb{K}_n(X \times \mathbb{A}^r) = \mathbb{K}_n(X)$ *for* $n \leq -d$.

Non-connective K theory of connective ring spectra

Theorem (Blumberg-Gepner-Tabuada)

Let R be a connective ring spetrum then

$$K_n(R) = K_n(\pi_0(R)), n \leq 0.$$

Definition

A functor $\mathbb{E} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{E}$ (presentable stable) is called a **localizing invariant** iff 1) it commutes with filtered colimits and 2) preserves exactness.

Theorem (Blumberg-Gepner-Tabuada)

For any localizing invariant $\mathbb{E} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}p$ *there is a natural equivalence*

$$\mathrm{Nat}(\mathbb{K}, \mathbb{E}) \simeq \mathbb{E}(\mathcal{S}p^\omega).$$

$$\begin{array}{ccc} \Sigma^\infty(-)^\simeq & \xrightarrow{\alpha} & \mathbb{E} \\ & \searrow^{\chi} & \uparrow{\scriptstyle \overline{\alpha}}\ {\scriptstyle \exists!} \\ & & \mathbb{K} \end{array}$$

Theorem (Blumberg-Gepner-Tabuada)

THH is localizing (hence additive) and

$$\mathrm{Nat}(K, THH) \simeq THH(\mathcal{S}p^{\omega}) \simeq THH(\mathbb{S}) \simeq \mathbb{S}$$

Hence up to homotopy there is a $\mathbb{Z}$ worth of such natural transformations. The Denis trace map $K \to THH$ corresponds to 1.

Thank You!