K-THEORY OF PROJECTIVE BUNDLES & BLOW-UPS

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Abstract. These are notes of my talk at the *K-theory and Derived Algebraic Geometry* seminar. The talk was mostly based on [Kha18]. The goal is to give formulas for additive invariants of projective bundles and blow-ups over derived Artin stacks.

1. Introduction

We will recall some terminology from previous talks: Artin stacks, perfect complexes and additive invariants. We then introduce semi-orthogonal decompositions, which are a tool for computing additive invariants.

After this, we will introduce projective bundles in the derived setting, give a semi-orthogonal decomposition on the ∞-category of quasi-coherent sheaves on a given projective bundle, and give a formula for computing the additive invariant of a projective bundle. This formula is part of the first main result of this talk (Theorem B).

The second main result (Theorem C) gives a formula for computing, in the derived setting, the additive invariant of a blow-up in a quasi-smooth center. In the third part of the talk we will first introduce blow-ups, using virtual Cartier divisors, and then say a few words about the proof of Theorem C

1.1. **Main results.** Let us first state the two main results in full. Of course, the necessary terminology will be introduced or recalled later on.

Theorem B. *Let X be a derived Artin stack, and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank $n+1$ with projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$. Then:*

1. *For each $0 \leq k \leq n$ we have a fully faithful functor*
$$\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E})) : \mathcal{F} \mapsto q^*\mathcal{F} \otimes \mathcal{O}(-k)$$
whose essential image we denote $\mathcal{C}(-k)$;
2. *The sequence of full subcategories $(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$ forms a semi-orthogonal decomposition of $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$;*
3. *For any additive invariant E of stable ∞-categories it holds $E(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0\leq m\leq n} E(X)$.*

Theorem C. *Let X be a derived Artin stacks, $i : Z \to X$ a quasi-smooth closed immersion of virtual codimension $n \geq 1$, and $p : \mathrm{Bl}_Z X \to X$ the blow-up of X in Z with exceptional divisor $Z \xleftarrow{q} \mathbb{P}(\mathcal{N}_{Z/X}) \xrightarrow{j} \mathrm{Bl}_Z X$.*

1. *We have a fully faithful functor $\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X) : \mathcal{F} \mapsto p^*\mathcal{F}$ whose essential image we denote $\mathcal{B}(0)$;*
2. *For $1 \leq k \leq n-1$, we have a fully faithful functor*
$$\mathcal{P}\mathrm{erf}(Z) \to \mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X) : \mathcal{F} \mapsto j_*(q^*\mathcal{F} \otimes \mathcal{O}(-k))$$
whose essential image we denote $\mathcal{B}(-k)$;

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(3) *The sequence of full subcategories* $(\mathcal{B}(0), \ldots, \mathcal{B}(-n+1))$ *forms a semi-orthogonal decomposition of* $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$*;*

(4) *For any additive invariant* E *of stable* ∞*-categories it holds* $E(\mathrm{Bl}_Z X) \simeq E(X) \oplus \bigoplus_{1 \leq k \leq n-1} E(Z)$.

1.2. **Recollections.**

1.2.1. *Artin stacks.*

Convention. From hereon, everything is derived. This means that categories are ∞-categories, tensor products are derived tensor products, sheafs are sheafs in the homotopical sense, etc..

Definition. We recall the following terminology from Stefan's and Eric's talks:

- A *stack* is a sheaf $X : \mathrm{s}\mathcal{R}\mathrm{ing} \to \mathcal{S}$ for the étale topology.
- A *scheme* is a stack X which is Zariski-locally of the form $\operatorname{Spec} R$ for $R \in \mathrm{s}\mathcal{R}\mathrm{ing}$.
- An *Artin stack* is a stack X which has an $f : U \to X$ with U a derived scheme and f smooth, epic and n-algebraic.

Throughout, let X be an Artin stack.

1.2.2. *Perfect complexes.*

Definition. We saw the following notions in Asaf's talk:

- For $R \in \mathrm{s}\mathcal{R}\mathrm{ing}$, we let $\mathcal{P}\mathrm{erf}(R)$ be the smallest stable subcategory of $\mathcal{M}\mathrm{od}(R)$ containing R which is closed under retracts.
- An $\mathcal{O}_X$-module $\mathcal{F} \in \mathrm{QCoh}(X)$ is *perfect* if $f^*\mathcal{F} \in \mathcal{P}\mathrm{erf}(R)$ for all $f : \operatorname{Spec} R \to X$. We let $\mathcal{P}\mathrm{erf}(X)$ be the full subcategory of $\mathrm{QCoh}(X)$ spanned by perfect $\mathcal{O}_X$-modules.

Proposition. *The category* $\mathrm{QCoh}(X)$ *is symmetric-monoidal, with a tensor product* $\otimes$ *that locally corresponds to the tensor product of rings. Now* $\mathcal{F} \in \mathrm{QCoh}(X)$ *is dualizable iff it is perfect.*

For $f : X \to Y$ a map of Artin stacks, we have an adjunction

$$f^* : \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*$$

where locally f^* is given by tensoring with $\Gamma(\mathcal{O}_Y) \to \Gamma(\mathcal{O}_X)$.

Facts. We have:

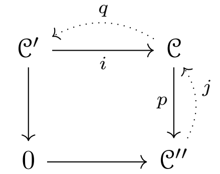
- The functor f^* restricts to a functor $\mathcal{P}\mathrm{erf}(Y) \to \mathcal{P}\mathrm{erf}(X)$;
- If f_* preserves colimits, then $f^* \dashv f_*$ satisfies the *projection formula*: for all $\mathcal{F} \in \mathrm{QCoh}(X), \mathcal{G} \in \mathrm{QCoh}(Y)$, the map

$$f_*(\mathcal{F}) \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^*\mathcal{G})$$

induced by transposing $f^*(f_*(\mathcal{F}) \otimes \mathcal{G}) \simeq f^*f_*\mathcal{F} \otimes f^*\mathcal{G} \to \mathcal{F} \otimes f^*\mathcal{G}$ is an equivalence [BFN10, Prop. 3.10].

1.2.3. *Additive invariants.*

Definition. A *split short exact sequence* of small stable categories is a bicartesian (solid) square



in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$, such that i, p have right adjoints $i \dashv q$ and $p \dashv j$ such that the unit $\mathrm{id}_{\mathcal{C}'} \to qi$ and the counit $pj \to id_{\mathcal{C}''}$ are invertible.

Observe that for the purpose of this talk, following [Kha18], we do not require additive invariants to commute with filtered colimits, as is done in [BGT13]

$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ is the category of small stable categories and exact functors between them.

Definition. Let $\mathcal{A}$ be a stable presentable category. An *additive invariant* is a functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \mathcal{A}$ such that for any split short exact sequence

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}''$$

the induced map $E(\mathcal{C}') \oplus E(\mathcal{C}'') \xrightarrow{(i,j)} E(\mathcal{C})$ is invertible, where $p \dashv j$.

Example. Algebraic K-theory $K : \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \mathcal{S}\mathrm{pc}$ is an additive invariant.

Definition. We define $E(X) \coloneqq E(\mathcal{P}\mathrm{erf}(X))$.

1.3. **Semi-orthogonal decompositions.**

Definition. Let $\mathcal{C}$ be a stable category with full stable subcategory $\mathcal{D}$.

- The category of objects *left orthogonal to* $\mathcal{D}$ is the full subcategory ${}^{\perp}\mathcal{D} \coloneqq \{x \in \mathcal{C} \mid \forall d \in \mathcal{D} : \mathcal{C}(x, d) \simeq *\}$
- The category of objects *right orthogonal to* $\mathcal{D}$ is the full subcategory $\mathcal{D}^{\perp} \coloneqq \{x \in \mathcal{C} \mid \forall d \in \mathcal{D} : \mathcal{C}(d, x) \simeq *\}$

Definition. Let $\mathcal{C}$ be stable. A *semi-orthogonal decomposition* of $\mathcal{C}$ is a sequence $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ of full stable subcategories such that

- For all integers $i > j$ it holds $\mathcal{C}(i) \subset {}^{\perp}\mathcal{C}(j)$;
- $\mathcal{C}$ is generated by $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ under finite limits and finite colimits.

Example. Let $\mathcal{C}$ be stable. The following two pieces of data are equivalent:

(1) A semi-orthogonal decomposition $(\mathcal{C}(0), \mathcal{C}(-1))$;
(2) A split short exact sequence $\mathcal{C}(0) \to \mathcal{C} \to \mathcal{C}(-1)$.

We will show how to go from (1) to (2). So let $(\mathcal{C}(0), \mathcal{C}(-1))$ be a semi-orthogonal decomposition of $\mathcal{C}$. Then for all $x \in \mathcal{C}$, we have a short exact sequence

$$x(0) \to x \to x(-1)$$

with $x(0) \in \mathcal{C}(0)$ and $x(-1) \in \mathcal{C}(-1)$. To see this, let $\mathcal{C}'$ be the full subcategory of all $x \in \mathcal{C}$ for which such a short exact sequence exists. Then $\mathcal{C}'$ is closed under finite limits and finite colimits, and contains $\mathcal{C}(0) \cup \mathcal{C}(-1)$. Hence $\mathcal{C}' = \mathcal{C}$.

One show that the above actually gives functors

$$q : \mathcal{C} \to \mathcal{C}(0) : x \mapsto x(0); \qquad \& \qquad p : \mathcal{C} \to \mathcal{C}(-1) : x \mapsto x(-1)$$

which induces a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(0) & \underset{f}{\overset{q}{\rightleftarrows}} & \mathcal{C} \\ \downarrow & & p \downarrow \uparrow r \\ 0 & \longrightarrow & \mathcal{C}(-1) \end{array}$$

such that $f \dashv q$ and $p \dashv r$. We claim this is a split short exact sequence.

In Kristian's talk, we saw that it suffices to check three things: f is fully faithful, the image of f is closed under retracts, and p is a localization of $\mathcal{C}$ with respect to the set of maps whose cofiber is in $\mathcal{C}(0)$. The first point is immediate.

For the second point, let $x \xrightarrow{u} x(0) \xrightarrow{v} x$ be given, such that $vu = \mathrm{id}$. Then consider the diagram

$$\begin{array}{ccccccc}
x(0) & \xrightarrow{v} & x & \xrightarrow{u} & x(0) & \xrightarrow{v} & x \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & x(-1) & \dashrightarrow & 0 & \longrightarrow & x(-1)
\end{array}$$

constructed as follows. One first takes the cofiber of v, call it $x(-1)$. One observes that $x(-1) \in \mathcal{C}(-1)$. This extends the solid diagram to the dotted diagram. The dashed arrow makes the diagram commute, since 0 is terminal.

Now the left square and the outer square is a pushout. Hence the rectangle made up from the middle and the right square is a pushout. Since $vu = \mathrm{id}$, it follows that the zero map on $x(-1)$ is the identity, hence $x \simeq x(0)$.

The third point follows from the adjunction $p \dashv r$, with r fully faithful.

Definition. Let $\mathcal{C}$ be stable and $S \subset \mathcal{C}$ a sub-simplicial set. Then we let $\mathrm{span}(S)$ be the smallest full subcategory of $\mathcal{C}$ containing S.

Lemma. *Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$. For $0 \leq m \leq n$, define $\mathcal{C}_{\leq -m} := \mathrm{span}(\mathcal{C}(-m) \cup \cdots \cup \mathcal{C}(-n))$ and put $\mathcal{C}_{\leq -n-1} := \{0\}$. Then there are split short exact sequences*

$$\mathcal{C}(-m) \to \mathcal{C}_{\leq -m} \to \mathcal{C}_{\leq -m-1}$$

for each $0 \leq m \leq n$.

Proof. By the previous example, it suffices to show that $(\mathcal{C}(-m), \mathcal{C}_{-m-1})$ is a semi-orthogonal decomposition of $\mathcal{C}_{\leq -m}$.

From $\mathcal{C}(-m) \subset {}^{\perp}\mathcal{C}(-m-1), \ldots, {}^{\perp}\mathcal{C}(-n)$, one concludes that $\mathcal{C}(-m) \subset \mathcal{C}_{\leq -m-1}$, since $\mathcal{C}_{\leq -m-1}$ is built from $\mathcal{C}(-m-1) \cup \cdots \cup \mathcal{C}(-n)$ using limits and shifts. And clearly $\mathcal{C}_{\leq m} = \mathrm{span}(\mathcal{C}(-m), \mathcal{C}_{\leq -m-1})$. The claim follows. □

Recall one formulation of the additivity theorem for K-theory: for any split short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of stable categories, the induced map $K(\mathcal{A}) \times K(\mathcal{C}) \to K(\mathcal{B})$ is an equivalence. The following can be seen as a generalization of the additivity theorem.

Lemma. *Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$. For E an additive invariant, it holds*

$$E(\mathcal{C}) \simeq \bigoplus_{0 \leq m \leq n} E(\mathcal{C}(-m))$$

Proof. By the previous lemma, we have a split short exact sequence

$$\mathcal{C}(0) \to \mathcal{C}_{\leq 0} = \mathcal{C} \to \mathcal{C}_{\leq -1}$$

which gives us

$$E(\mathcal{C}) \simeq E(\mathcal{C}(0)) \oplus E(\mathcal{C}_{\leq -1})$$

by definition of E being an additive invariant. Now $\mathcal{C}(-1), \ldots, \mathcal{C}(-n)$ is a semi-orthogonal decomposition of $\mathcal{C}_{\leq -1}$, so induction gives us

$$E(\mathcal{C}) \simeq E(\mathcal{C}(0)) \oplus \bigoplus_{1 \leq m \leq n} E(\mathcal{C}(-m))$$

and the claim follows. □

2. The projective bundle formula

2.1. Projective bundles.

Definition. A *locally free module* $\mathcal{E}$ *of rank* n on X is an $\mathcal{O}_X$-module $\mathcal{E} \in \mathrm{QCoh}(X)$ such that for all $x : \mathrm{Spec}\, A \to X$ it holds that $x^*\mathcal{E}$ is locally free of rank n on $\mathrm{Spec}\, A$, i.e. Zariski-locally of the form A^n.

When $n = 1$, we call $\mathcal{E}$ an *invertible sheaf* or *line bundle*. The category of such is written $\mathrm{Pic}(X)$.

Observe that invertible sheafs are dualizable, hence perfect.

Definition. Let $\mathcal{E} \in \mathrm{QCoh}(X)$ be locally free of rank n. The *projective bundle* associated to $\mathcal{E}$ is a derived stack $q : \mathbb{P}(\mathcal{E}) \to X$ such that for all $x : S \to X$ in $\mathrm{d}\mathcal{S}\mathrm{ch}_{/X}$ we have that $\mathbb{P}(\mathcal{E})(S)$ is the space of pairs $(\mathcal{L}, u)$ with $\mathcal{L} \in \mathrm{Pic}(S)$ and $u : x^*(\mathcal{E}) \to \mathcal{L}$ a morphism of $\mathcal{O}_S$-modules, surjective on π_0.

This universal property induces an invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ together with a surjection $\pi : q^*\mathcal{E} \to \mathcal{O}(1)$. For any $y : U \to \mathbb{P}(\mathcal{E})$ with $U \in \mathrm{d}\mathcal{S}\mathrm{ch}_{/X}$, the map $y^*\pi : y^*q^*\mathcal{E} \to y^*\mathcal{O}(1)$ is the surjection corresponding to the map $U \to \mathbb{P}(\mathcal{E})$.

Definition. For $k \geq 0$ we put $\mathcal{O}(k) \coloneqq \mathcal{O}(1)^{\otimes k}$. Since $\mathcal{O}(1)$ is invertible, we have a sheaf $\mathcal{O}(-1)$ such that $\mathcal{O}(1) \otimes \mathcal{O}(-1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$. For $k \leq 0$, we put $\mathcal{O}(k) \coloneqq \mathcal{O}(-1)^{\otimes(-k)}$. We then have $\mathcal{O}(n) \otimes \mathcal{O}(m) \simeq \mathcal{O}(n+m)$ for all $n, m \in \mathbb{Z}$.

From hereon, fix a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $n+1$, and let $q : \mathbb{P}(\mathcal{E}) \to X$ be the corresponding projective bundle.

2.2. Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$.

We will now sketch some arguments that go into proving Theorem B. We will consider the three parts separately.

We will use Serre's formula:

Proposition (Serre). *We have* $q_*(\mathcal{O}(0)) \simeq \mathcal{O}_X$ *and* $q_*(\mathcal{O}(-m)) \simeq 0$ *for* $1 \leq m \leq n$.

Theorem B.1. *For each* $0 \leq k \leq n$ *we have a fully faithful functor*

$$\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E})) : \mathcal{F} \mapsto q^*\mathcal{F} \otimes \mathcal{O}(-k)$$

Proof. We first show that we have a fully faithful functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathbb{P}(\mathcal{E}))$: $\mathcal{F} \mapsto q^*\mathcal{F} \otimes \mathcal{O}(-k)$. Since $\mathcal{O}(-k)$ is invertible, it suffices to do the case $k = 0$. We want the unit map $\mathcal{F} \to q_*q^*\mathcal{F}$ to be invertible. This is local, so we reduce to the case where $X = \mathrm{Spec}\, R$ and $\mathcal{E} = \mathcal{O}_X^{n+1}$. In this case, q^*, q_* both commute with colimits, and $\mathrm{QCoh}(X) = \mathcal{M}\mathrm{od}_R$ is generated by R under colimits and finite limits. We may thus assume that $\mathcal{F} = \mathcal{O}_X$. Now the claim follows from Serre's formula.

Now it suffices to show that $\mathcal{F} \mapsto q^*\mathcal{F} \otimes \mathcal{O}(-k)$ restricts to perfect complexes. And indeed this is so: q^* preserves perfect complexes; $\mathcal{O}(-k)$ is a line bundle, hence perfect; $\mathcal{P}\mathrm{erf}(Y)$ is stable under tensor products, for any Y. □

Definition. For any k, let $\mathcal{C}(-k)$ be the essential image of the functor in (B.1).

Theorem B. *The categories* $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ *form a semi-orthogonal decomposition of* $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$.

We recall the following:

Facts. It holds

- Projection formula: $q_*(\mathcal{F}) \otimes \mathcal{G} \simeq q_*(\mathcal{F} \otimes q^*\mathcal{G})$;
- $q^* \dashv q_*$ restricts to $q^* : \mathcal{P}\mathrm{erf}(X) \rightleftarrows \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E})) : q_*$;
- $(-) \otimes \mathcal{O}(n)$ gives a functor $\mathcal{C}(-k) \to \mathcal{C}(-k+n)$.

Theorem B.2. *The categories* $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ *form a semi-orthogonal decomposition of* $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$.

Proof. We can write $\mathcal{C}(i)$ as $\{q^*\mathcal{F} \otimes \mathcal{O}(i) \mid \mathcal{F} \in \mathcal{P}\mathrm{erf}(X)\}$. We first check that $\mathcal{C}(i) \subset {}^{\perp}\mathcal{C}(j)$ for $0 \geq i > j \geq -n$. To this end, observe that for all m it holds:

$$\begin{aligned}\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))(q^*\mathcal{F}, q^*\mathcal{G} \otimes \mathcal{O}(-m)) &\simeq \mathcal{P}\mathrm{erf}(X)(\mathcal{F}, q_*(q^*\mathcal{G} \otimes \mathcal{O}(-m))) \\ &\simeq \mathcal{P}\mathrm{erf}(X)(\mathcal{F}, \mathcal{G} \otimes q_*\mathcal{O}(-m))\end{aligned}$$

by the projection formula. Hence, by Serre's formula, it holds

$$\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))(q^*\mathcal{F}, q^*\mathcal{G} \otimes \mathcal{O}(-m)) \simeq 0$$

for all $1 \leq m \leq n$. Taking m such that $-m + i = j$, gives us

$$\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))(q^*\mathcal{F} \otimes \mathcal{O}(i), q^*\mathcal{G} \otimes \mathcal{O}(j)) \simeq 0$$

which was to be shown.

Now let $\mathcal{F} \in \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$ be given. We sketch the argument why $\mathcal{F}$ is in $\mathrm{span}(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$. Put $\mathcal{G}_{-1} := \mathcal{F} \otimes \mathcal{O}(-1)$. For $m \geq 0$, define $\mathcal{G}_m$ via the following cofiber sequence:

$$q^*q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1)) \to \mathcal{G}_{m-1} \otimes \mathcal{O}(1) \to \mathcal{G}_m$$

One first shows that $\mathcal{G}_n \simeq 0$, which is some work. Since $q^*q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1)) \in \mathcal{C}(0)$, it follows that $\mathcal{G}_{n-1} \in \mathcal{C}(-1)$. In general, in the cofiber sequence

$$\mathcal{G}_{n-i-1} \otimes \mathcal{O}(1) \to \mathcal{G}_{n-i} \to q^*q_*(\mathcal{G}_{n-i-1} \otimes \mathcal{O}(1))[1]$$

the last term is in $\mathcal{C}(0)$, and by induction the middle term is in $\mathrm{span}(\mathcal{C}(-1), \ldots, \mathcal{C}(-i))$. It follows that the first term is in $\mathrm{span}(\mathcal{C}(0), \ldots, \mathcal{C}(-i))$. In particular, $\mathcal{G}_{-1} \otimes \mathcal{O}(1) \simeq \mathcal{F}$ is in $\mathrm{span}(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$, which is what we wanted. □

Theorem B.3. *For any additive invariant* E *it holds* $E(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0 \leq m \leq n} E(X)$.

Proof. The semi-orthogonal decomposition $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ of $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$ gives

$$E(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0 \leq m \leq n} E(\mathcal{C}(-m))$$

Now $(-) \otimes \mathcal{O}(-m) : \mathcal{C}(0) \simeq \mathcal{C}(-m)$ gives us

$$E(\mathcal{C}(-m)) \simeq E(\mathcal{C}(0)) \simeq E(X)$$

since $\mathcal{C}(0)$ is the essential image of $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E}))$. □

3. The blow-up formula

3.1. Virtual Cartier Divisors.

Convention. For the purpose of the rest of these notes, we assume X is a scheme. But everything generalizes to X an Artin stack.

Definition. A *closed immersion of schemes* is a map $i : Z \to X$ of schemes such that $Z_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is a closed immersion in the classical sense, i.e. Zariski-locally on X, i is of the form $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $A \to B$ is surjective on π_0.

From hereon, fix a closed immersion $Z \to X$.

Definition. We say i is *quasi-smooth of virtual codimension* n if Zariski-locally on X, there exists a morphism $f : X \to \mathbb{A}^n$ and a Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^n \end{array}$$

in $\mathrm{d}\mathcal{S}\mathrm{ch}$.

In other words, i is quasi-smooth if Zariski-locally on X, it corresponds to a map of rings $A \to A/(f_1, \dots, f_n) \coloneqq A \otimes_{\mathbb{Z}[T_1,\dots,T_n]} \mathbb{Z}[T_1, \dots, T_n]/(T_1, \dots, T_n)$.

Example. If X, Z are classical, then i is quasi-smooth iff it is Koszul-regular, i.e. iff Z is locally cut out by a Koszul-regular sequence $(f_1, \dots, f_n)$, i.e. for which the Koszul-complex $A/(f_1, \dots, f_n)$ is discrete.

Definition. A *virtual Cartier divisor* on X is a quasi-smooth closed immersion $D \to X$ of virtual codimension 1.

Construction.

- Recall the *cotangent complex* $\mathcal{L}_{U/V}$ associated to a map of schemes $U \to V$ is the derived analogue of the Kähler differentials.
- For any quasi-smooth closed immersion $j : W \to Y$, define the *conormal sheaf* of j as $\mathcal{N}_{W/Y} \coloneqq \mathcal{L}_{W/Y}[-1]$.
- The conormal sheaf of j is locally free of rank the virtual codimemsion of j.

From hereon, $i : Z \to X$ is quasi-smooth.

Definition. For any $f : S \to X$ in $\mathrm{d}\mathcal{S}\mathrm{ch}$, a *virtual Cartier divisor on S over* (X, Z) is a commutative square in $\mathrm{d}\mathcal{S}\mathrm{ch}$

$$\begin{array}{ccc} D & \longrightarrow & S \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ Z & \longrightarrow & X \end{array}$$

such that

(1) D is a virtual Cartier divisor on S;
(2) The underlying square of classical schemes is cartesian;
(3) The canonical map $g^*\mathcal{N}_{Z/X} \to \mathcal{N}_{D/S}$ is surjective.

3.2. Blow-ups.

Definition. For $S \to X$ in $\mathrm{d}\mathcal{S}\mathrm{ch}$, let $\mathrm{Bl}_Z(X)(S \to X)$ be the space of virtual Cartier divisors on S over (X, Z). This gives a functor

$$\mathrm{Bl}_Z X : \mathrm{d}\mathcal{S}\mathrm{ch}^{\mathrm{op}}_{/X} \to \mathcal{S} : (S \to X) \mapsto \mathrm{Bl}_Z(X)(S \to X)$$

Theorem ([KR18])**.** *The functor* $\mathrm{Bl}_Z X$ *is a scheme.*

Example. If X, Z are classical, then so is $\mathrm{Bl}_Z X$, and it coincides with the classical blow-up of X in Z.

Remark (The universal virtual Cartier divisor)**.** The universal property of $\mathrm{Bl}_Z X$ gives us a virtual Cartier divisor on $\mathrm{Bl}_Z X$ over (X, Z) as follows:

$$\begin{array}{ccc} D & \xrightarrow{\ j\ } & \mathrm{Bl}_Z X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

It is a result that D is a projective bundle. In fact, $D = \mathbb{P}(\mathcal{N}_{Z/X})$.

3.3. Semi-orthogonal decomposition on $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$.

Theorem C. *Under the standing assumptions:*

(1) We have a fully faithful functor $\mathcal{P}\mathrm{erf}(X) \to \mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X) : \mathcal{F} \mapsto p^*\mathcal{F}$
(2) For $1 \leq k \leq n-1$*, we have a fully faithful functor*

$$\mathcal{P}\mathrm{erf}(Z) \to \mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X) : \mathcal{F} \mapsto j_*(q^*\mathcal{F} \otimes \mathcal{O}(-k))$$

whose essential image we denote $\mathcal{B}(-k)$*;*

(3) *The sequence of full subcategories* $(\mathcal{B}(0), \dots, \mathcal{B}(-n+1))$ *forms a semi-orthogonal decomposition of* $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$*;*
(4) *For any additive invariant* E *of stable* ∞*-categories it holds* $E(\mathrm{Bl}_Z X) \simeq E(X) \oplus \bigoplus_{1 \le k \le n-1} E(Z)$.

We will sketch some arguments.

Proof of (1). The idea is again to show that

$$\mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Bl}_Z X) : \mathcal{F} \mapsto p^* \mathcal{F}$$

is fully faithful, since $\mathcal{F} \to p_* p^* \mathcal{F}$ is invertible. This is local, so we assume that $X = \operatorname{Spec} R$ and that $Z = \operatorname{Spec} R/(f_1, \dots, f_n)$ for some points $f_i \in R$.

Now $\mathrm{QCoh}(X)$ is generated by $\mathcal{O}_X$ under colimits and finite limits, and p_* commutes with colimits, so we can take $\mathcal{F} = \mathcal{O}_X$. Then $\mathcal{O}_X \to p_* \mathcal{O}_{\mathrm{Bl}_Z X}$ is a base change of the map $\varphi : \mathcal{O}_{\mathcal{A}^n} \to p'_* \mathcal{O}_{\mathrm{Bl}_{\{0\}} \mathbb{A}^n}$, for the blowup $p' : \mathrm{Bl}_{\{0\}} \mathbb{A}^n \to \mathbb{A}^n$ of affine space in the origin. Now φ is an isomorphism, which is a classical statement found in [SGA6, Exp. VII]. □

Proof of (2). The idea is as follows. We have an adjunction

$$j_* : \mathrm{QCoh}(D) \rightleftarrows \mathrm{QCoh}(\mathrm{Bl}_Z X) : j^!$$

which gives us $j_* q^* \dashv q_* j^!$. It suffices to show that $\varphi_{\mathcal{F}} : \mathcal{F} \to q_* j^! j_* q^* \mathcal{F}$ is invertible. One again reduces to the case where X is affine, and $\mathcal{F} = \mathcal{O}_Z$.

One shows that $\varphi_{\mathcal{O}_Z}$ is equivalent to the map

$$\mathcal{O}_Z \to q_*(\mathcal{O}_D(-1)) \oplus q_* \mathcal{O}_D$$

Which is an equivalence by Serre's formula.

Now one restricts the fully faithful functor $\mathrm{QCoh}(Z) \to \mathrm{QCoh}(\mathrm{Bl}_Z X)$ to perfect complezes. □

Proof of (3). We will indicate why $\mathrm{span}(\mathcal{B}(0), \dots, \mathcal{B}(-n+1))$ is all of $\mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$. Let $\mathcal{F} \in \mathcal{P}\mathrm{erf}(\mathrm{Bl}_Z X)$ be given. Take a short exact sequence

$$p^* p_* \mathcal{F} \to \mathcal{F} \to \mathcal{G}_0$$

Define $\mathcal{G}_m$ recursively via the short exact sequences

$$j_*(q^* q_*(j^!(\mathcal{G}_{m-1}) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(-m)) \to \mathcal{G}_{m-1} \to \mathcal{G}_m$$

where the first arrow is induced by the counit of $j_* q^* \dashv q_* j^!$.

One shows that $\mathcal{G}_{n-1} \simeq 0$, which is some work. Then, by a similar induction as before, one concludes that we have an short exact sequence

$$j_*(q^* q_*(j^!(\mathcal{G}_1) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(-m)) \to \mathcal{G}_0 \to \mathcal{G}_1$$

with the first term in $\mathcal{B}(-1)$, the last term in $\mathrm{span}(\mathcal{B}(0), \dots, \mathcal{B}(-n+1))$, hence the middle term in $\mathrm{span}(\mathcal{B}(0), \dots, \mathcal{B}(-n+1))$. □

Proof of (4). The semi-orthogonal decomposition gives

$$E(\mathrm{Bl}_Z X) \simeq \bigoplus_{0 \ge m \ge -n+1} (\mathcal{B}(m))$$

Now use points (1) and (2) to get the desired formula. □

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